The Gołab-Schinzel Functional Equation Restricted to Half-Lines

By

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Abstract

Under some regularity conditions, we determine solutions of the functional equation
\[ f(x + yf(x)) = f(x)f(y), \quad x, y \in \mathbb{R}, \]
on domains restricted to half-lines.

Key words and phrases: Gołab-Schinzel functional equation, restricted domain, iterative functional equation, Euler equation, system of iterative equations, convex solution, monotonic solution, one-to-one solution.

1. Introduction

The Gołab-Schinzel functional equation
\[ f(x + yf(x)) = f(x)f(y), \quad x, y \in \mathbb{R}, \]
had appeared in connection with determining some subgroups or subsemigroups of the affine group (ST. GOŁAB and A. SCHINZEL [7], cf. also J. ACZÉL and J. DHOMBRES [1], Chapter 19), and was considered by many authors (cf. for instance [2], [4], [5], [8]). It is known that the nontrivial continuous solution \( f: \mathbb{R} \rightarrow \mathbb{R} \) is either of the form
\[ f(x) = cx + 1, \quad \text{or} \quad f(x) = \max(cx + 1, 0), \quad x \in \mathbb{R}, \]
with a real constant $c$. Recently, motivated by some applications [9],
the solutions of the Gołąb-Schinzel equation on restricted domains
like $\{(x,y): x \geq 0, y \geq 0\}$ or $\{(x,y): x > 0, y > 0\}$ have been
determined ([3], [6], [12], [13], [14]).

The present paper is concerned with a stronger limitation of the
variables in the Gołąb-Schinzel equation. Restricting this equation to
the half-lines
\[ \{(x,y): x = 1, \ y = t, \ t > 0\} \quad \text{and} \quad \{(x,y): x = t, \ y = 1, \ t > 0\}, \]
and setting
\[ b := f(1), \]
we obtain two functional equations in a single variable (of iterative type)
\[ f(1 + bt) = bf(t), \]  
\[ \quad (1) \]
and
\[ f(t + f(t)) = bf(t). \]  
\[ \quad (2) \]
Equation (1) is a classical homogeneous linear equation (cf. [10], pp.
58, 106), and equation (2) is a composite equation which for $b = 1$
becomes the well-known Euler equation related to invariant curves
(cf. [10], p. 286).

In Section 2 we consider equation (1). Using the general solution,
we show that, under some regularity assumptions, the solution of
equation (1) is proportional to the basic solution of the original
Gołąb-Schinzel functional equation.

In Section 3 we deal with equation (2). It turns out that, under some
regularity conditions, the solution of equation (2) is the sum of the
basic solution of the Gołąb-Schinzel equation and a constant.

The most interesting result is given in Section 4. Assume that $b > 1$
and $a > 0$ are fixed. Applying iterative methods, we show that if a
function $f: (a, \infty) \to (0, \infty)$ is one-to-one in a neighbourhood of
infinity and satisfies both equations (1) and (2), then $f(t) = (b - 1)t + 1$
for all $t > a$. A generalization of this result is also given.

2. Linear Equation

We begin with the following

**Theorem 1.** Let $b > 0$, $b \neq 1$, and let
\[ a := \frac{1}{1 - b} \quad \text{if} \ b < 1, \]
and let $a \geq \frac{1}{1 - b}$ be arbitrarily fixed if $b > 1$. 
A function $f: (a, \infty) \to \mathbb{R}$ satisfies the equation
\[ f(1 + bt) = bf(t), \quad t > a, \quad (3) \]
iff there is a log $b$-periodic function $p: \mathbb{R} \to \mathbb{R}$ such that
\[ f(t) = [(b - 1)t + 1]p\left(\log \left(t - \frac{1}{b - 1}\right)\right) \quad \text{for } t > a; \]
moreover, $f$ is positive (nonnegative) iff so is $p$.

**Proof.** Assume first that the function $f: (a, \infty) \to \mathbb{R}$ is a solution of equation (3). Since $f_0: (a, \infty) \to \mathbb{R}$,
\[ f_0(t) := (b - 1)t + 1 \]
satisfies equation (3), the function $g: (a, \infty) \to \mathbb{R}$ given by
\[ g := \frac{f}{f_0} \]
satisfies the functional equation
\[ g(1 + bt) = g(t), \quad t > a. \]
(The particular solution $f_0$ of equation (3) in the case $0 < b < 1$ is negative and in the case $b > 1$ is positive on the interval under consideration, hence $g = f/f_0$ can be formed.) Writing the latter equation in the form
\[ g\left(\exp\left(\log \left(t + \frac{1}{b - 1}\right) + \log b\right) - \frac{1}{b - 1}\right) = g\left(\exp\left(\log \left(t + \frac{1}{b - 1}\right)\right) - \frac{1}{b - 1}\right), \]
we infer that the function $p$ defined by
\[ p(u) := g\left(\exp u - \frac{1}{b - 1}\right) \]
is log $b$-periodic, that is,
\[ p(u + \log b) = p(u). \]
Since $f(t) = f_0(t)g(t)$, we hence obtain
\[ f(t) = [(b - 1)t + 1]p\left(\log t - \frac{1}{b - 1}\right), \quad t > a. \]
It is easy to verify that for an arbitrary log \( b \)-periodic function \( p: \mathbb{R} \to \mathbb{R} \), the function \( f \) given by this formula is a solution of equation (3) in \((a, \infty)\). This completes the proof.

**Remark 1.** For a fixed \( t_0 > \frac{1}{1-b} \) and for every real function \( f_0 \) defined on the closed interval with endpoints \( t_0 \) and \( 1 + bt_0 \) and such that \( f_0(1 + bt_0) = bf_0(t_0) \), there is a unique solution \( f: (\frac{1}{1-b}, \infty) \to \mathbb{R} \) of equation (3) which is an extension of \( f_0 \). Moreover, if \( f_0 \) is continuous or (and) monotonic then so is \( f \). Thus the continuous and monotonic solution of equation (3) depends on an arbitrary function (cf. M. Kuczma [10], e.g. sect. III. 4 and sect. V. 3).

**Remark 2.** For \( b = 1 \) equation (3) becomes an equation for \( 1 \)-periodic functions.

As a simple consequence of Theorem 1 we obtain the following

**Corollary 1.** Let \( b \in (0, 1) \) and \( a = \frac{1}{1-b} \) be fixed. If \( f: (a, \infty) \to [0, \infty) \) is a decreasing solution of equation (3), then \( f = 0 \) in \((a, \infty)\).

**Theorem 2.** Let \( b > 1 \) and \( a \geq \max(0, \frac{1}{1-b}) \) be fixed. Suppose that \( f: (a, \infty) \to \mathbb{R} \) is a solution of equation (3). If the function

\[
(a, \infty) \ni t \to \frac{f(t)}{t},
\]

is monotonic in \((\alpha, \infty)\) for some \( \alpha > a \), or there exists a finite

\[
\lim_{t \to \infty} \frac{f(t)}{t},
\]

then there is a \( c \in \mathbb{R} \), such that

\[
f(t) = c[(b-1)t + 1], \quad t > a.
\]

**Proof.** By Theorem 1 there is a log \( b \)-periodic function such that

\[
\frac{f(t)}{t} = \left( (b-1) + \frac{1}{t} \right) p\left( \log t - \frac{1}{b-1} \right), \quad t > a.
\]

Since \( \lim_{t \to \infty} ((b-1) + \frac{1}{t}) = b-1 \), the assumed monotonicity of this function implies that \( p \) must be constant. The remaining part is obvious. This completes the proof.

Applying some well-known properties of convex functions we hence obtain

**Corollary 2.** Let \( b > 1 \) and \( a \geq \max(0, \frac{1}{1-b}) \) be fixed. If \( f: (a, \infty) \to \mathbb{R} \) is a solution of equation (3) and it is convex or
concave in an interval \((\alpha, \infty)\) for some \(\alpha > a\), then there is a \(c \in \mathbb{R}\) such that
\[
f(t) = c[(b - 1)t + 1], \quad t > a.
\]

**Proof.** Suppose that \(f\) is convex in an interval \((\alpha, \infty)\) for some \(\alpha > a\). By the well-known properties of convex functions, the one-sided derivatives \(f'_-\) and \(f'_+\) exist in \((\alpha, \infty)\) and are nondecreasing. In view of Theorem 1 we have
\[
f(t) = [(b - 1)t + 1]p\left(\log t - \frac{1}{b - 1}\right), \quad t > a,
\]
where \(p\) is a log \(b\)-periodic function. It follows that \(p'_-\) and \(p'_+\) exist too. Denoting by \(p'\) one of these one-sided derivatives we infer that
\[
f'(t) = (b - 1)p\left(\log t - \frac{1}{b - 1}\right) + \frac{(b - 1)t + 1}{t}p'\left(\log t - \frac{1}{b - 1}\right)
\]
is nondecreasing in \((\alpha, \infty)\). Replacing here \(t\) by \(e^{t+n\log b}\) and taking into account that \(p\) and \(p'\) are log \(b\)-periodic, we hence obtain that, for every positive integer \(n\), the function
\[
f'(e^{t+n\log b}) = (b - 1)p\left(t - \frac{1}{b - 1}\right) + \frac{(b - 1)e^{t+n\log b} + 1}{e^{t+n\log b}}p'\left(t - \frac{1}{b - 1}\right)
\]
is nondecreasing in \((\alpha, \infty)\). Letting \(n \to \infty\) we infer that the function \((b - 1)(p + p')\) is nondecreasing in \((\alpha, \infty)\). The periodicity of \(p\) and \(p'\) implies that \(p + p'\) is constant. Thus there is a \(c \in \mathbb{R}\) such that
\[
p + p' = c.
\]
Since \(p'\) denotes a one-sided derivative, it follows that \(p'_- = p'_+\) and, consequently, \(p\) is differentiable. Solving the above differential equation for \(p\) and taking into account that \(p\) is periodic we conclude that \(p\) is constant. This completes the proof.

**3. Composite Equation**

In this section we deal with the composite functional equation (2). We begin with the following

**Theorem 3.** Let \(b \in (-1, 1)\), \(b \neq 0\), be fixed. If \(f: \mathbb{R} \to \mathbb{R}\) is a continuous solution of the functional equation
\[
f(t + f(t)) = bf(t), \quad t \in \mathbb{R},
\]
possessing only one zero, then there is a \( c \in \mathbb{R} \) such that

\[
f(t) = (b - 1)t + c, \quad t \in \mathbb{R}.
\]

**Proof.** From (2), by induction, we obtain

\[
f\left(t + \frac{1 - b^n}{1 - b} f(t)\right) = b^n f(t), \quad t \in \mathbb{R},
\]

for all positive integers \( n \). Letting \( n \to \infty \), and taking into account that \( f \) is continuous and \( |b| < 1 \), we get

\[
f\left(t + \frac{1}{1 - b} f(t)\right) = 0, \quad t \in \mathbb{R}.
\]

Denoting by \( t_0 \) the only zero of the function \( f \), we have

\[
t + \frac{1}{1 - b} f(t) = t_0, \quad t \in \mathbb{R}.
\]

Setting \( c := (1 - b)t_0 \) we obtain the desired formula.

In the case \( b = 1 \) equation (2) becomes the well-known Euler functional equation. Recall the following (cf. M. Kuczma [10], p. 286)

**Theorem 4 (Kuratowski, Wagner).** The only solutions \( f: \mathbb{R} \to \mathbb{R} \) of the functional equation

\[
f(t + f(t)) = f(t), \quad t \in \mathbb{R},
\]

possessing the Darboux property, are constant functions.

In the case \( b > 1 \) we have the following

**Theorem 5.** Let \( b > 1 \), and \( a > 0 \) be fixed. If \( f: (a, \infty) \to [0, \infty) \) is a solution of the functional equation

\[
f(t + f(t)) = bf(t), \quad t > a,
\]

and \( f \) is convex or concave in \((\alpha, \infty)\) for some \( \alpha > a \), then there are \( c \in \mathbb{R} \) and \( \beta \geq \alpha \) such that

\[
f(t) = (b - 1)t + c, \quad t > \beta.
\]

If moreover \( f: (a, \infty) \to (0, \infty) \) then

\[
f(t) = (b - 1)t + c, \quad t > a.
\]

**Proof.** By induction, for all positive integers \( n \), we have

\[
f\left(t + \frac{b^n - 1}{b - 1} f(t)\right) = b^n f(t), \quad t > a.
\]
If $f = 0$ in $(a, \infty)$ then there is nothing to prove. In the opposite case there is a $t_0 > a$ such that $f(t_0) > 0$. Putting

$$t_n := t_0 + \frac{b^n - 1}{b - 1}f(t_0), \quad n \in \mathbb{N},$$

we get

$$f(t_n) = b^n f(t_0), \quad n \in \mathbb{N}.$$

Consequently,

$$\frac{f(t_{n+1}) - f(t_n)}{t_{n+1} - t_n} = \frac{b^{n+1}f(t_0) - b^n f(t_0)}{(t_0 + \frac{b^{n+1}-1}{b-1}f(t_0)) - (t_0 + \frac{b^n-1}{b-1}f(t_0))}, \quad n \in \mathbb{N},$$

which simplifies to the relation

$$\frac{f(t_{n+1}) - f(t_n)}{t_{n+1} - t_n} = b - 1, \quad n \in \mathbb{N}.$$

Since

$$\lim_{n \to \infty} t_n = \infty,$$

the convexity (or concavity) of $f$ in $(\alpha, \infty)$ implies that there is a $\beta \geq \alpha$ such that

$$\frac{f(t) - f(s)}{t - s} = b - 1, \quad s, t \in (\beta, \infty).$$

It follows that there is a $c \in \mathbb{R}$ such that

$$f(t) = (b - 1)t + c, \quad t > \beta. \quad (5)$$

If $f$ is positive then for every $t > a$ there is a positive integer $n$ such that

$$t + \frac{b^n - 1}{b - 1}f(t) > \beta.$$

Now from (4) and (5) we get

$$(b - 1)\left( t + \frac{b^n - 1}{b - 1}f(t) \right) + c = b^n f(t)$$

which simplifies to the desired relation

$$f(t) = (b - 1)t + c.$$

This completes the proof.
Theorem 6. Let $b > 0$, $b \neq 1$, be fixed. Suppose that either $f: [0, \infty) \to [f(0), \alpha]$ where $\alpha \leq \infty$, or $f: [0, \infty) \to (\alpha, f(0)]$ where $\alpha \geq -\infty$, is a bijective solution of the functional equation
\[ f(t + f(t)) = bf(t), \quad t \geq 0. \]
If $f^{-1}$ is continuous at $f(0)$ then
\[ f(t) = (b - 1)t + f(0), \quad t \geq 0. \]

Proof. Suppose that $f: [0, \infty) \to [f(0), \alpha]$. The function $g := f^{-1}$ satisfies the functional equation
\[ g(bs) = g(s) + s, \quad s \geq f(0), \]
and is right continuous at $f(0)$. Applying Theorem 5.1 in M. Kuczma [10], p. 106, we obtain
\[ g(s) = f(0) + \frac{s}{b - 1}, \quad s \geq f(0). \]
Since an argument to show the remaining statement is analogous, this completes the proof.

Similarly, applying Theorem 5.3 in M. Kuczma [10], p. 108, we can prove

Theorem 7. Let $b > 0$, $b \neq 1$, be fixed. Suppose that $f: (0, \infty) \to (\alpha, \beta)$ where $-\infty \leq \alpha < \beta \leq \infty$. If $f$ is a homeomorphic solution of the functional equation
\[ f(t + f(t)) = bf(t), \quad t > 0, \]
then
\[ f(t) = (b - 1)t + f(0), \quad t > 0. \]

4. A System of Functional Equations

From the point of view of the theory of the Gołąb-Schinzel functional equation the most interesting result of this paper reads as follows.

Theorem 8. Let $b > 1$, and $a > 0$ be fixed. Suppose that $f: (a, \infty) \to (0, \infty)$ satisfies the pair of functional equations
\[ f(1 + bt) = bf(t), \quad f(t + f(t)) = bf(t), \quad t > a. \quad (6) \]
If $f$ is one-to-one in a neighbourhood of $\infty$, then
\[ f(t) = (b - 1)t + 1, \quad t > a. \]
Proof. Suppose that \( f: (a, \infty) \to (0, \infty) \) satisfying system (6) is one-to-one in an interval \((\beta, \infty)\) for some \( \beta \geq a \). Iterating the first equation we obtain
\[
f\left(\sum_{k=0}^{n} b^k + b^{n+1} t\right) = b^{n+1} f(t), \quad t > a, \ n \in \mathbb{N}.
\]
Similarly, iterating the second equation, we get
\[
f\left(t + \left(\sum_{k=0}^{n} b^k\right)f(t)\right) = b^{n+1} f(t), \quad t > a, \ n \in \mathbb{N}.
\]
Both these equations imply that
\[
f\left(\sum_{k=0}^{n} b^k + b^{n+1} t\right) = f\left(t + \left(\sum_{k=0}^{n} b^k\right)f(t)\right), \quad t > a, \ n \in \mathbb{N}. \tag{7}
\]
Take an arbitrary \( t > a \). Since \( f(t) > 0 \) and \( b > 1 \) there is a positive integer \( n \) such that
\[
\sum_{k=0}^{n} b^k + b^{n+1} t > \beta, \quad t + \left(\sum_{k=0}^{n} b^k\right)f(t) > \beta.
\]
Now the injectivity of \( f \) in \((\beta, \infty)\) and relation (7) imply that
\[
\sum_{k=0}^{n} b^k + b^{n+1} t = t + \left(\sum_{k=0}^{n} b^k\right)f(t),
\]
which simplifies to the relation
\[
f(t) = (b - 1)t + 1,
\]
as desired. The proof is completed.

Now, replacing the injectivity condition by a much weaker one, we prove a generalization of the previous result.

**Theorem 9.** Let \( b > 1 \), and \( a > 0 \) be fixed. Suppose that \( f: (a, \infty) \to (0, \infty) \) satisfies the pair of functional equations (6). If there exist \( \alpha \geq a \) and \( M > 0 \) such that, for all \( t_1, t_2 > \alpha \),
\[
f(t_1) = f(t_2) \Rightarrow |t_1 - t_2| \leq M,
\]
then
\[
f(t) = (b - 1)t + 1, \quad t > a.
\]
Proof. Let us fix arbitrarily $t > a$ and put

$$t_{1,n} := \sum_{k=0}^{n} b^k + b^{n+1} t, \quad t_{2,n} := t + \left( \sum_{k=0}^{n} b^k \right) f(t).$$

According to (7) we have

$$f(t_{1,n}) = f(t_{2,n}), \quad n \in \mathbb{N}.$$  

As

$$\lim_{n \to \infty} t_{1,n} = \infty = \lim_{n \to \infty} t_{2,n},$$

for sufficiently large $n$, we have $t_{1,n}, t_{2,n} > \alpha$. From the assumed implication we obtain that, for all sufficiently large $n \in \mathbb{N}$,

$$|t_{2,n} - t_{1,n}| = \frac{b^{n+1} - 1}{b - 1} |f(t) - 1 - (b - 1)t| \leq M.$$

Since $\lim_{n \to \infty} b^{n+1} = \infty$, we infer that

$$f(t) - 1 - (b - 1)t = 0,$$

which completes the proof.

5. Final Remarks

Restricting the Gołąb-Schinzel functional equation to the pair of half-lines $\{(x, y): x = a, y > 0\}$ and $\{(x, y): x > 0, y = a\}$ where $a > 0$ is fixed, and assuming $b := f(a)$, one gets the pair of iterative functional equations

$$f(a + bt) = bf(t), \quad f(t + af(t)) = bf(t), \quad t > 0.$$

The iteration procedure applied to each of these equations leads to two infinite systems of equations,

$$f\left( \sum_{k=0}^{n} b^k + b^{n+1} t \right) = b^{n+1} f(t), \quad t > a, \quad n \in \mathbb{N},$$

and

$$f\left( t + \left( \sum_{k=0}^{n} b^k \right) f(t) \right) = b^{n+1} f(t), \quad t > a, \quad n \in \mathbb{N}.$$  

Using these systems one can prove analogous results as above.

Let us mention that the system of iterative functional equations which appears when the Gołąb-Schinzel functional equation is
restricted to the two parallel lines \( x = a \) and \( x = b \) was considered in [11].

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**References**


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