Electronic Transactions on Numerical Analysis.
Volume 51, pp. 262–273, 2019.

ISSN 1068–9613.
DOI: 10.1553/etna_vol51s262

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${}^{\gamma}_{\delta}\mathbf{\Phi}$-TYPE INCLUSION SET FOR EIGENVALUES OF A TENSOR*

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Abstract. In this paper, a new ${}^{\gamma}_{\delta}\mathbf{\Phi}$-type eigenvalue inclusion set for tensors is given, and some inclusion relations between this new inclusion set and other ones are presented. In addition, a new sufficient criterion for identifying nonsingular tensors is also provided by using the new ${}^{\gamma}_{\delta}\mathbf{\Phi}$-type eigenvalue inclusion set. Some numerical results are reported to show the superiority of the results.

Key words. tensor, eigenvalue, inclusion, nonsingular, ${}^{\gamma}_{\delta}\mathbf{\Phi}$-type.

AMS subject classifications. 15A69, 15A18, 65F15, 65H17, 15A15, 65F40

1. Introduction.

We first recall some definitions for tensors. $\mathcal{A} = (a_{i_1\cdots i_m})_{\boldsymbol{n}}$ is called a tensor of order m and dimension $\boldsymbol{n} = n_1 \times \cdots \times n_m$ over the field $\mathbb{F}$ if

$$\mathcal{A} = (a_{i_1\cdots i_m})_{\boldsymbol{n}} = (a_{i_1\cdots i_m})_{n_1\times\cdots\times n_m} \in \mathbb{F}^{[m,\boldsymbol{n}]} = \mathbb{F}^{n_1\times\cdots\times n_m}.$$

When $\mathbb{F} = \mathbb{C}$, $\mathcal{A}$ is called a complex tensor; when $\mathbb{F} = \mathbb{R}$, $\mathcal{A}$ is called a real tensor; when $n_1 = \cdots = n_m = n$, $\mathcal{A}$ is simply called a tensor of order m and dimension n over the field $\mathbb{F}$, and we denote $\mathbb{F}^{[m,\boldsymbol{n}]}$ by $\mathbb{F}^{[m,n]}$ if there is no danger of confusion. If the entries $a_{i_1\cdots i_m}$ are invariant under any permutation of their indices, then $\mathcal{A}$ is called a symmetric tensor.

In 2005, Qi [14] and Lim [12] independently introduced the notion of eigenvalues of tensors. For $\mathcal{A} = (a_{i_1\cdots i_m})_{n\times\cdots\times n} \in \mathbb{C}^{[m,n]}$, $x = (x_1, \ldots, x_n)^\top \in \mathbb{C}^n$, $\mathcal{A}x^{m-1}$ is a dimension n column vector with entries

$$\left(\mathcal{A}x^{m-1}\right)_i = \sum_{(i_2,\ldots,i_m)\in N^{m-1}} a_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m}, \qquad i \in N = \{1, \ldots, n\}.$$

If there exists a nonzero vector $x = (x_1, \ldots, x_n)^\top \in \mathbb{C}^n$ and a number $\lambda \in \mathbb{C}$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called an eigenvalue of $\mathcal{A}$ and x is called an eigenvector of $\mathcal{A}$ corresponding to λ, where

$$x^{[m-1]} = (x_1^{m-1}, \ldots, x_n^{m-1})^\top.$$

Let $\sigma(\mathcal{A})$ denote the set of all eigenvalues of $\mathcal{A}$, and $\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}$ be the spectral radius of $\mathcal{A}$. A complex tensor $\mathcal{A}$ is called nonsingular if $0 \notin \sigma(\mathcal{A})$, otherwise it is called singular.

In recent years, the spectral theory of tensors has attracted much attention [7]. Although the eigenvalues of tensors have many applications in numerical multilinear algebra [13, 18, 19], their computation is, like most tensor problems, NP-hard [4]. Hence efficient algorithms to (approximately) locate all eigenvalues of a given tensor have become increasingly important.

In 2005, Qi [14] gave a Geršgorin-type eigenvalue inclusion set for a real symmetric tensor $\mathcal{A} = (a_{i_1\cdots i_m})$ in the following form:

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) = \bigcup_{i\in N} \Gamma_i(\mathcal{A}),$$

*Received March 9, 2019. Accepted July 4, 2019. Published online on September 10, 2019. Recommended by Wen-Wei Lin.

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where

$$\Gamma_i(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{i\cdots i}| \le R_i(D_i)\}, \qquad D_i = N^{m-1}\backslash\{(i,\ldots,i)\},$$

$$R_i(E) = \sum_{(j_2,\ldots,j_m)\in E} |a_{ij_2\cdots j_m}|, \qquad \forall E \subseteq N^{m-1}.$$

This result also holds for $\mathcal{A} \in \mathbb{C}^{[m,n]}$ [10, 18]. In 2014, Li et al. [10] and in 2016, Li et al. [9] gave two variations of Brauer-type eigenvalue inclusion sets for a tensor $\mathcal{A} = (a_{i_1\cdots i_m})$ as follows:

$$\sigma(\mathcal{A}) \subseteq \Phi(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}),$$

where

$$\begin{aligned}
\Phi(\mathcal{A}) &= \bigcup_{(i,j)\in N\times N_i} \Phi_{ij}(\mathcal{A}),\\
\Phi_{ij}(\mathcal{A}) &= \{z \in \mathbb{C} : (|z - a_{i\cdots i}| - R_i(S_i))|z - a_{j\cdots j}| \le R_i(N_i^{m-1})R_j(D_j)\},\\
S_i &= \{(j_2,\ldots,j_m) \in N^{m-1} : i \in \{j_2,\ldots,j_m\} \ne \{i\}\},\\
\mathcal{K}(\mathcal{A}) &= \bigcup_{(i,j)\in N\times N_i} \mathcal{K}_{ij}(\mathcal{A}),\\
\mathcal{K}_{ij}(\mathcal{A}) &= \{z \in \mathbb{C} : (|z - a_{i\cdots i}| - R_i(D_{ij}))|z - a_{j\cdots j}| \le |a_{ij\cdots j}|R_j(D_j)\},\\
D_{ij} &= D_i\backslash\{(j,\ldots,j)\}, \qquad N_i = N\backslash\{i\}.
\end{aligned}$$

In 2017, Sang et al. [15] gave another variation of Brauer-type eigenvalue inclusion sets for a tensor $\mathcal{A} = (a_{i_1\cdots i_m})$:

$$\sigma(\mathcal{A}) \subseteq \Omega(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}),$$

where

$$\begin{aligned}
\Omega(\mathcal{A}) &= \bigcup_{(i,j)\in N\times N_i} \Omega_{ij}(\mathcal{A}),\\
\Omega_{ij}(\mathcal{A}) &= \{z \in \mathbb{C} : (|z - a_{i\cdots i}| - R_i(D_i\backslash\omega_i))|z - a_{j\cdots j}| \le R_i(\omega_i)R_j(D_j)\},\\
\omega_i &= \{(k,\ldots,k) \in N^{m-1} : k \in N_i\}.
\end{aligned}$$

In addition, several other eigenvalue inclusion sets for tensors were derived in [2, 6, 7, 8, 10, 11, 15], and relations between some of them were given.

In this paper, we introduce a new eigenvalue inclusion set, $^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A})$, for a tensor $\mathcal{A}$. Moreover, the inclusion relation between $^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A})$ and other eigenvalue inclusion sets is discussed. As an application, a new criterion for identifying nonsingular tensors [14, 17] is provided. In order to show the superiority of the new results, numerical examples are given in Section 3.

2. A new $^{\gamma}_{\delta}\mathbf{\Phi}$-type eigenvalue inclusion set for tensors. In this section, we first establish a new $^{\gamma}_{\delta}\mathbf{\Phi}$-type eigenvalue inclusion set for tensors, then point out some relations between several eigenvalue inclusion sets including the $^{\gamma}_{\delta}\mathbf{\Phi}$-type one, followed up with a new sufficient condition for a tensor to be nonsingular.

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The main theorem of this section reads as follows:

THEOREM 2.1. *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$ *with* $m, n \geq 2$. *Then*

$$\sigma(\mathcal{A}) \subseteq {}_{\delta}^{\gamma}\boldsymbol{\Phi}(\mathcal{A}) = \bigcup_{(i,j)\in N\times N_i} {}_{\delta}^{\gamma}\boldsymbol{\Phi}_{ij}(\mathcal{A}),$$

where

$$\begin{aligned}
{}_{\delta}^{\gamma}\boldsymbol{\Phi}_{ij}(\mathcal{A}) &= {}_{\delta}^{\gamma}\Phi_{ij}(\mathcal{A}) \cup {}^{\delta}\Gamma_{ji}(\mathcal{A}),\\
{}_{\delta}^{\gamma}\Phi_{ij}(\mathcal{A}) &= \Big\{ z \in \mathbb{C} : (|z - a_{i\cdots i}| - R_i(D_i\backslash\Gamma_{ij}))(|z - a_{j\cdots j}| - R_j(D_j\backslash\Delta_{ij}))\\
&\qquad\qquad \leq R_i(\Gamma_{ij})R_j(\Delta_{ij}) \Big\},\\
{}^{\delta}\Gamma_{ji}(\mathcal{A}) &= \{z \in \mathbb{C} : |z - a_{j\cdots j}| \leq R_j(D_j\backslash\Delta_{ij})\},\\
\Gamma_{ij} &= N_i \times \gamma_{ij}, \quad \Delta_{ij} = (L_i \cup (N_i \times \delta_{ij}))\backslash\{(j,\ldots,j)\},\\
\gamma_{ij} &\subseteq N_i^{m-2}, \ \ \delta_{ij} \subseteq N_i^{m-2},\\
L_i &= \{(j_2,\ldots,j_m) \in N^{m-1} : i \in \{j_2,\ldots,j_m\}\}, \qquad N_i = N\backslash\{i\},\\
{}_{\delta}^{\gamma}\boldsymbol{\Phi}(\mathcal{A}) &= {}_{\delta}^{\gamma}\Phi(\mathcal{A}) \cup {}^{\delta}\Gamma(\mathcal{A}),\\
{}_{\delta}^{\gamma}\Phi(\mathcal{A}) &= \bigcup_{(i,j)\in N\times N_i} {}_{\delta}^{\gamma}\Phi_{ij}(\mathcal{A}), \qquad {}^{\delta}\Gamma(\mathcal{A}) = \bigcup_{(i,j)\in N\times N_i} {}^{\delta}\Gamma_{ji}(\mathcal{A}).
\end{aligned}$$

Proof. Let $\lambda \in \sigma(\mathcal{A})$ and $x = (x_1,\ldots,x_n)^\top \in \mathbb{C}^n\backslash\{0\}$ be an associated eigenvector, namely,

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{2.1}$$

Let $|x_{\mu_1}| \geq |x_{\mu_2}| \geq \cdots \geq |x_{\mu_n}|$. Then $|x_{\mu_1}| \neq 0$.

From (2.1), we have

$$\begin{aligned}
(\lambda - a_{\mu_1\cdots\mu_1})x_{\mu_1}^{m-1} = &\sum_{(i_2,\ldots,i_m)\in\Gamma_{\mu_1\mu_2}} a_{\mu_1 i_2\cdots i_m} x_{i_2}\cdots x_{i_m}\\
&+ \sum_{(i_2,\ldots,i_m)\in D_{\mu_1}\backslash\Gamma_{\mu_1\mu_2}} a_{\mu_1 i_2\cdots i_m} x_{i_2}\cdots x_{i_m},
\end{aligned}$$

hence,

$$\begin{aligned}
|\lambda - a_{\mu_1\cdots\mu_1}||x_{\mu_1}|^{m-1} \leq &\sum_{(i_2,\ldots,i_m)\in\Gamma_{\mu_1\mu_2}} |a_{\mu_1 i_2\cdots i_m}||x_{i_2}|\cdots|x_{i_m}|\\
&+ \sum_{(i_2,\ldots,i_m)\in D_{\mu_1}\backslash\Gamma_{\mu_1\mu_2}} |a_{\mu_1 i_2\cdots i_m}||x_{i_2}|\cdots|x_{i_m}|\\
\leq &\sum_{(i_2,\ldots,i_m)\in\Gamma_{\mu_1\mu_2}} |a_{\mu_1 i_2\cdots i_m}||x_{\mu_2}|^{m-1}\\
&+ \sum_{(i_2,\ldots,i_m)\in D_{\mu_1}\backslash\Gamma_{\mu_1\mu_2}} |a_{\mu_1 i_2\cdots i_m}||x_{\mu_1}|^{m-1}\\
= &R_{\mu_1}(\Gamma_{\mu_1\mu_2})|x_{\mu_2}|^{m-1} + R_{\mu_1}(D_{\mu_1}\backslash\Gamma_{\mu_1\mu_2})|x_{\mu_1}|^{m-1},
\end{aligned}$$

which is equivalent to

$$(|\lambda - a_{\mu_1\cdots\mu_1}| - R_{\mu_1}(D_{\mu_1}\backslash\Gamma_{\mu_1\mu_2}))|x_{\mu_1}|^{m-1} \le R_{\mu_1}(\Gamma_{\mu_1\mu_2})|x_{\mu_2}|^{m-1}. \tag{2.2}$$

If $|x_{\mu_2}| = 0$, then $|\lambda - a_{\mu_1\cdots\mu_1}| - R_{\mu_1}(D_{\mu_1}\backslash\Gamma_{\mu_1\mu_2}) \le 0$ as $|x_{\mu_1}| > 0$, and it is obvious that

$$\lambda \in {}^{\gamma}_{\delta}\mathbf{\Phi}_{\mu_1\mu_2}(\mathcal{A}) = {}^{\gamma}_{\delta}\Phi_{\mu_1\mu_2}(\mathcal{A}) \cup {}^{\delta}\Gamma_{\mu_2\mu_1}(\mathcal{A}) \subseteq {}^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A}) = {}^{\gamma}_{\delta}\Phi(\mathcal{A}) \cup {}^{\delta}\Gamma(\mathcal{A}).$$

If $|x_{\mu_2}| > 0$, then from (2.1), we obtain

$$(|\lambda - a_{\mu_2\cdots\mu_2}| - R_{\mu_2}(D_{\mu_2}\backslash\Delta_{\mu_1\mu_2}))|x_{\mu_2}|^{m-1} \le R_{\mu_2}(\Delta_{\mu_1\mu_2})|x_{\mu_1}|^{m-1}. \tag{2.3}$$

If $|\lambda - a_{\mu_2\cdots\mu_2}| - R_{\mu_2}(D_{\mu_2}\backslash\Delta_{\mu_1\mu_2}) \le 0$, then $\lambda \in {}^{\delta}\Gamma_{\mu_2\mu_1}(\mathcal{A}) \subseteq {}^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A})$. If, on the other hand, $|\lambda - a_{\mu_2\cdots\mu_2}| - R_{\mu_2}(D_{\mu_2}\backslash\Delta_{\mu_1\mu_2}) > 0$, by multiplying (2.2) with (2.3), we get

$$\begin{aligned}&(|\lambda - a_{\mu_1\cdots\mu_1}| - R_{\mu_1}(D_{\mu_1}\backslash\Gamma_{\mu_1\mu_2})) \times \\ &\qquad\qquad (|\lambda - a_{\mu_2\cdots\mu_2}| - R_{\mu_2}(D_{\mu_2}\backslash\Delta_{\mu_1\mu_2}))\,|x_{\mu_1}|^{m-1}|x_{\mu_2}|^{m-1} \\ &\quad \le R_{\mu_1}(\Gamma_{\mu_1\mu_2})R_{\mu_2}(\Delta_{\mu_1\mu_2})|x_{\mu_2}|^{m-1}|x_{\mu_1}|^{m-1}.\end{aligned}$$

Note that $|x_{\mu_1}|^{m-1}|x_{\mu_2}|^{m-1} > 0$. Then

$$\begin{aligned}&(|\lambda - a_{\mu_1\cdots\mu_1}| - R_{\mu_1}(D_{\mu_1}\backslash\Gamma_{\mu_1\mu_2}))\,(|\lambda - a_{\mu_2\cdots\mu_2}| - R_{\mu_2}(D_{\mu_2}\backslash\Delta_{\mu_1\mu_2})) \\ &\qquad \le R_{\mu_1}(\Gamma_{\mu_1\mu_2})R_{\mu_2}(\Delta_{\mu_1\mu_2}).\end{aligned}$$

This implies $\lambda \in {}^{\gamma}_{\delta}\Phi_{\mu_1\mu_2}(\mathcal{A}) \subseteq {}^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A})$. Therefore, $\sigma(A) \subseteq {}^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A})$. □

REMARK 2.2. (i) The set ${}^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A})$ in Theorem 2.1 is called a ${}^{\gamma}_{\delta}\mathbf{\Phi}$-region of $\mathcal{A}$ or a (γ, δ)-doubly diagonally inclusion set ((γ, δ)-DDIS) of $\mathcal{A}$.
(ii) If $\delta_{ij} = N_i^{m-2}$ for all $(i, j) \in N \times N_i$, then ${}^{\gamma}_{\delta}\mathbf{\Phi}_{ij}(\mathcal{A}) = {}^{\gamma}_{\delta}\Phi_{ij}(\mathcal{A})$ due to the fact that

$$\Delta_{ij} = (L_i \cup (N_i \times \delta_{ij}))\backslash\{(j, \ldots, j)\} = D_j,$$

which implies

$$R_j(D_j\backslash\Delta_{ij}) = 0.$$

This means

$${}^{\delta}\Gamma_{ji}(\mathcal{A}) = \{a_{j\cdots j}\} \subseteq {}^{\gamma}_{\delta}\Phi_{ij}(\mathcal{A}).$$

In this case, we denote ${}^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A})$, ${}^{\gamma}_{\delta}\mathbf{\Phi}_{ij}(\mathcal{A})$ by ${}^{\gamma}\Phi(\mathcal{A})$, ${}^{\gamma}\Phi_{ij}(\mathcal{A})$, respectively.
(iii) If $\gamma_{ij} = N_i^{m-2}$ for all $(i, j) \in N \times N_i$, we denote ${}^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A})$, ${}^{\gamma}_{\delta}\mathbf{\Phi}_{ij}(\mathcal{A})$ by ${}_{\delta}\mathbf{\Phi}(\mathcal{A})$, ${}_{\delta}\mathbf{\Phi}_{ij}(\mathcal{A})$, respectively.
(iv) If $\gamma_{ij} = N_i^{m-2}$,$\delta_{ij} = N_i^{m-2}$ for all $(i, j) \in N \times N_i$, we denote ${}^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A})$, ${}^{\gamma}_{\delta}\mathbf{\Phi}_{ij}(\mathcal{A})$ by $\Phi(\mathcal{A})$, $\Phi_{ij}(\mathcal{A})$, respectively.
(v) If $m = 2$, noting that

$$\Gamma_{ij} = N_i \times \gamma_{ij} = N_i = D_i, \quad \Delta_{ij} = (L_i \cup (N_i \times \delta_{ij}))\backslash\{(j, \ldots, j)\} = N_j = D_j,$$

then the set ${}^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A})$ reduces to the Brauer set of matrices; see [1].

From Theorem 2.1, several corollaries follow. As shown below, the inclusion sets $\Phi(\mathcal{A})$ in [9, Theorem 2.1], $\Gamma(\mathcal{A})$ in [7, 10, 14, 18], $\Theta(\mathcal{A})$ in Corollary 2.5 below, and $\Omega(\mathcal{A})$ in [15, Theorem 2.1] can be viewed as results of the application of Theorem 2.1 for special cases.

COROLLARY 2.3 ([9, Theorem 2.1]). *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$ *with* $m, n \geq 2$. *Then*

$$\sigma(\mathcal{A}) \subseteq \Phi(\mathcal{A}) = \bigcup_{(i,j)\in N\times N_i} \Phi_{ij}(\mathcal{A}),$$

where

$$\begin{aligned}\Phi_{ij}(\mathcal{A}) &= \{z \in \mathbb{C} : (|z - a_{i\cdots i}| - R_i(S_i))|z - a_{j\cdots j}| \leq R_i(N_i^{m-1})R_j(D_j)\},\\ S_i &= \{(j_2, \ldots, j_m) \in N^{m-1} : i \in \{j_2, \ldots, j_m\} \neq \{i\}\}.\end{aligned}$$

Proof. Let $\gamma_{ij} = N_i^{m-2}, \delta_{ij} = N_i^{m-2}$ for all $(i, j) \in N \times N_i$. From Theorem 2.1, the conclusion follows easily. ∎

COROLLARY 2.4 ([7, 10, 14, 18]). *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$ *with* $m \geq 2, n \geq 1$. *Then*

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) = \bigcup_{i\in N} \Gamma_i(\mathcal{A}),$$

where

$$\Gamma_i(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{i\cdots i}| \leq R_i(D_i)\}.$$

Proof. If $n = 1$, the conclusion is obviously correct. Now, assume $n > 1$. Let $\gamma_{ij} = \emptyset$ for all $(i, j) \in N \times N_i$. From Theorem 2.1, we have

$$\begin{aligned}&(|z - a_{i\cdots i}| - R_i(D_i))(|z - a_{j\cdots j}| - R_j(D_j\backslash\Delta_{ij})) \leq 0, \quad \text{i.e.,}\\ &|z - a_{j\cdots j}| \leq R_j(D_j\backslash\Delta_{ij}).\end{aligned}$$

Then

$$^{\gamma}_{\delta}\mathbf{\Phi}_{ij}(\mathcal{A}) = \Gamma_i(\mathcal{A}) \cup \{z \in \mathbb{C} : |z - a_{j\cdots j}| \leq R_j(D_j\backslash\Delta_{ij})\},$$

that is,

$$^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A}) = \Gamma(\mathcal{A}). \qquad ∎$$

COROLLARY 2.5. *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$ *with* $m, n \geq 2$. *Then*

$$\sigma(\mathcal{A}) \subseteq \Theta(\mathcal{A}) = \bigcup_{(i,j)\in N\times N_i} \Theta_{ij}(\mathcal{A}),$$

where

$$\begin{aligned}\Theta_{ij}(\mathcal{A}) &= \{z \in \mathbb{C} : (|z - a_{i\cdots i}| - R_i(D_i\backslash\theta_i))|z - a_{j\cdots j}| \leq R_i(\theta_i)R_j(D_j)\},\\ \theta_i &= N_i \times \{(k, \ldots, k) \in N^{m-2} : k \in N_i\}.\end{aligned}$$

Proof. Let $\gamma_{ij} = \{(k, \ldots, k) \in N^{m-2} : k \in N_i\}, \delta_{ij} = N_i^{m-2}$ for all $(i, j) \in N \times N_i$. From Theorem 2.1, we have

$$\begin{aligned}(|z - a_{i\cdots i}| - R_i(D_i\backslash\theta_i))\,|z - a_{j\cdots j}| &\leq R_i(\theta_i)R_j(D_j), \qquad \text{or,}\\ |z - a_{j\cdots j}| &\leq 0, \qquad \text{or equivalently,} \qquad z = a_{j\cdots j}.\end{aligned}$$

Then

$$^{\gamma}_{\delta}\mathbf{\Phi}_{ij}(\mathcal{A}) = \Theta_{ij}(\mathcal{A}) \cup \{a_{j\cdots j}\} = \Theta_{ij}(\mathcal{A}),$$

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that is,

$$ {}^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A}) = \Theta(\mathcal{A}). \qquad \square $$

COROLLARY 2.6 ([15, Theorem 2.1]). *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$ *with* $m, n \geq 2$. *Then*

$$ \sigma(\mathcal{A}) \subseteq \Omega(\mathcal{A}) = \bigcup_{(i,j)\in N\times N_i} \Omega_{ij}(\mathcal{A}), $$

where

$$ \begin{aligned} \Omega_{ij}(\mathcal{A}) &= \{z \in \mathbb{C} : (|z - a_{i\cdots i}| - R_i(D_i\backslash\omega_i))|z - a_{j\cdots j}| \leq R_i(\omega_i)R_j(D_j)\}, \\ \omega_i &= \{(k, \ldots, k) \in N^{m-1} : k \in N_i\}. \end{aligned} $$

Proof. Let $\Gamma_{ij} = \omega_i, \delta_{ij} = N_i^{m-2}$ for all $(i, j) \in N \times N_i$. From Theorem 2.1, we have

$$ \begin{aligned} (|z - a_{i\cdots i}| - R_i(D_i\backslash\omega_i))\,|z - a_{j\cdots j}| \leq R_i(\omega_i)R_j(D_j), \qquad \text{or,} \\ |z - a_{j\cdots j}| \leq 0, \quad \text{ or equivalently, } \quad z = a_{j\cdots j}. \end{aligned} $$

Then

$$ {}^{\gamma}_{\delta}\mathbf{\Phi}_{ij}(\mathcal{A}) = \Omega_{ij}(\mathcal{A}) \cup \{a_{j\cdots j}\} = \Omega_{ij}(\mathcal{A}), $$

that is,

$$ {}^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A}) = \Omega(\mathcal{A}). \qquad \square $$

The next proposition shows that the eigenvalue inclusion sets ${}^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A})$ in Theorem 2.1 and $\Gamma(\mathcal{A})$ [10, 14, 18] in Corollary 2.4 have an inclusion relationship.

PROPOSITION 2.7. *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$ *with* $m, n \geq 2$. *Then*

$$ {}^{\gamma}_{\delta}\mathbf{\Phi}_{ij}(\mathcal{A}) \subseteq \Gamma_i(\mathcal{A}) \cup \Gamma_j(\mathcal{A}). $$

Hence,

$$ {}^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}). $$

Proof. Let $z \in {}^{\gamma}_{\delta}\mathbf{\Phi}_{ij}(\mathcal{A}) = {}^{\gamma}_{\delta}\Phi_{ij}(\mathcal{A}) \cup {}^{\delta}\Gamma_{ji}(\mathcal{A})$. Then z satisfies

$$ \begin{aligned} &(|z - a_{i\cdots i}| - R_i(D_i\backslash\Gamma_{ij}))(|z - a_{j\cdots j}| - R_j(D_j\backslash\Delta_{ij})) \leq R_i(\Gamma_{ij})R_j(\Delta_{ij}), \quad \text{ or} \\ &|z - a_{j\cdots j}| \leq R_j(D_j\backslash\Delta_{ij}). \end{aligned} $$

- If $|z - a_{j\cdots j}| \leq R_j(D_j\backslash\Delta_{ij})$, then $z \in \Gamma_j(\mathcal{A})$.
- If $|z - a_{j\cdots j}| > R_j(D_j\backslash\Delta_{ij})$, then $z \in {}^{\gamma}_{\delta}\Phi_{ij}(\mathcal{A})$.
- If $R_i(\Gamma_{ij})R_j(\Delta_{ij}) = 0$, then $|z - a_{i\cdots i}| - R_i(D_i\backslash\Gamma_{ij}) \leq 0$, consequently, $z \in \Gamma_i(\mathcal{A})$.

Now, assume that $R_i(\Gamma_{ij})R_j(\Delta_{ij}) > 0$.

- If $|z - a_{i\cdots i}| \leq R_i(D_i\backslash\Gamma_{ij})$, then $z \in \Gamma_i(\mathcal{A})$.
- If $|z - a_{i\cdots i}| > R_i(D_i\backslash\Gamma_{ij})$, then, from $z \in {}^{\gamma}_{\delta}\Phi_{ij}(\mathcal{A})$, we have

$$ \frac{|z - a_{i\cdots i}| - R_i(D_i\backslash\Gamma_{ij})}{R_i(\Gamma_{ij})} \frac{|z - a_{j\cdots j}| - R_j(D_j\backslash\Delta_{ij})}{R_j(\Delta_{ij})} \leq 1. \tag{2.4} $$

Hence, from (2.4), we obtain

$$\frac{|z - a_{i\cdots i}| - R_i(D_i\backslash\Gamma_{ij})}{R_i(\Gamma_{ij})} \leq 1 \quad \text{or} \quad \frac{|z - a_{j\cdots j}| - R_j(D_j\backslash\Delta_{ij})}{R_j(\Delta_{ij})} \leq 1,$$

namely, $z \in \Gamma_i(\mathcal{A}) \cup \Gamma_j(\mathcal{A})$. Thus, ${}^{\gamma}_{\delta}\mathbf{\Phi}_{ij}(\mathcal{A}) \subseteq \Gamma_i(\mathcal{A}) \cup \Gamma_j(\mathcal{A})$. Hence, ${}^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$. The proof is completed. $\square$

To compare the sets $\Phi(\mathcal{A})$ [9, Theorem 2.1] in Corollary 2.3, ${}^{\gamma}_{\delta}\Phi(\mathcal{A})$, ${}^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A})$, ${}^{\gamma}_{\delta}\mathbf{\Phi}_{ij}(\mathcal{A})$ in Theorem 2.1, $\Theta(\mathcal{A})$ in Corollary 2.5, $\Omega(\mathcal{A})$ [15, Theorem 2.1] in Corollary 2.6, $\mathcal{K}(\mathcal{A})$ in [10, Theorem 2.1], and $\Gamma(\mathcal{A})$ [10, 14, 18] in Corollary 2.4, we need the following lemma provided in [9].

LEMMA 2.8 ([9, Lemmas 2.2 and 2.3]). *Let* $a, b, c \geq 0$, *and* $d > 0$.

(I) *If* $\frac{a}{b+c+d} \leq 1$, *then*

$$\frac{a-(b+c)}{d} \leq \frac{a-b}{c+d} \leq \frac{a}{b+c+d}.$$

(II) *If* $\frac{a}{b+c+d} \geq 1$, *then*

$$\frac{a-(b+c)}{d} \geq \frac{a-b}{c+d} \geq \frac{a}{b+c+d}.$$

Now, a comparison of ${}^{\gamma^1}_{\delta^1}\mathbf{\Phi}(\mathcal{A})$ and ${}^{\gamma^2}_{\delta^1}\mathbf{\Phi}(\mathcal{A})$ is established as follows.

PROPOSITION 2.9. *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$ *with* $m, n \geq 2$,

$$\begin{aligned}
&\Gamma^1_{ij} = N_i \times \gamma^1_{ij}, \ \ \Delta^1_{ij} = (L_i \cup (N_i \times \delta^1_{ij}))\backslash\{(j,\dots,j)\}, \ \ \gamma^1_{ij} \subseteq N_i^{m-2}, \ \ \delta^1_{ij} \subseteq N_i^{m-2},\\
&\Gamma^2_{ij} = N_i \times \gamma^2_{ij}, \ \ \Delta^2_{ij} = (L_i \cup (N_i \times \delta^2_{ij}))\backslash\{(j,\dots,j)\}, \ \ \gamma^2_{ij} \subseteq N_i^{m-2}, \ \ \delta^2_{ij} \subseteq N_i^{m-2},\\
&\Gamma^1_{ij} \supseteq \Gamma^2_{ij}, \ \ \Delta^1_{ji} \supseteq \Gamma^1_{ij}, \quad \text{and} \quad \Delta^1_{ij} \supseteq \Gamma^2_{ji},
\end{aligned}$$

for all $(i, j) \in N \times N_i$. *Then for all* $(i, j) \in N \times N_i$,

$${}^{\gamma^1}_{\delta^1}\mathbf{\Phi}_{ij}(\mathcal{A}) \subseteq {}^{\gamma^2}_{\delta^1}\mathbf{\Phi}_{ij}(\mathcal{A}) \cup {}^{\gamma^2}_{\delta^1}\mathbf{\Phi}_{ji}(\mathcal{A}).$$

Hence,

$${}^{\gamma^1}_{\delta^1}\mathbf{\Phi}(\mathcal{A}) \subseteq {}^{\gamma^2}_{\delta^1}\mathbf{\Phi}(\mathcal{A}).$$

Thus,

$${}^{\gamma^1}\Phi(\mathcal{A}) \subseteq {}^{\gamma^2}\Phi(\mathcal{A}), \qquad \Phi(\mathcal{A}) \subseteq \Theta(\mathcal{A}) \subseteq \Omega(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}).$$

Proof. Let $z \in {}^{\gamma^1}_{\delta^1}\mathbf{\Phi}_{ij}(\mathcal{A})$. Then either

$$(|z - a_{i\cdots i}| - R_i(D_i\backslash\Gamma^1_{ij}))(|z - a_{j\cdots j}| - R_j(D_j\backslash\Delta^1_{ij})) \leq R_i(\Gamma^1_{ij})R_j(\Delta^1_{ij}) \tag{2.5}$$

or

$$|z - a_{j\cdots j}| \leq R_j(D_j\backslash\Delta^1_{ij}),$$

is fulfilled.

- If $z \in {}^{\delta^1}\Gamma_{ji}(\mathcal{A})$, then $z \in {}^{\gamma^2}_{\delta^1}\mathbf{\Phi}_{ij}(\mathcal{A})$.
- If $z \notin {}^{\delta^1}\Gamma_{ji}(\mathcal{A})$, then $z \in {}^{\gamma^1}_{\delta^1}\Phi_{ij}(\mathcal{A})$.

• If $R_i(\Gamma_{ij}^1)R_j(\Delta_{ij}^1) = 0$, then by $z \notin {}^{\delta^1}\Gamma_{ji}(\mathcal{A})$, $\Gamma_{ij}^1 \supseteq \Gamma_{ij}^2$, and (2.5), we obtain

$$(|z - a_{i\cdots i}| - R_i(D_i \backslash \Gamma_{ij}^2))(|z - a_{j\cdots j}| - R_j(D_j \backslash \Delta_{ij}^1)) \le 0 \le R_i(\Gamma_{ij}^2)R_j(\Delta_{ij}^1),$$

which implies that $z \in {}^{\gamma^2}_{\delta^1}\Phi_{ij}(\mathcal{A})$.

• If $R_i(\Gamma_{ij}^1)R_j(\Delta_{ij}^1) > 0$, then from (2.5), we obtain

$$\frac{|z - a_{i\cdots i}| - R_i(D_i \backslash \Gamma_{ij}^1)}{R_i(\Gamma_{ij}^1)} \frac{|z - a_{j\cdots j}| - R_j(D_j \backslash \Delta_{ij}^1)}{R_j(\Delta_{ij}^1)} \le 1, \tag{2.6}$$

which implies

$$\frac{|z - a_{i\cdots i}| - R_i(D_i \backslash \Gamma_{ij}^1)}{R_i(\Gamma_{ij}^1)} \le 1, \tag{2.7}$$

or

$$\frac{|z - a_{j\cdots j}| - R_j(D_j \backslash \Delta_{ij}^1)}{R_j(\Delta_{ij}^1)} \le 1. \tag{2.8}$$

Let $a = |z - a_{i\cdots i}|, b = R_i(D_i \backslash \Gamma_{ij}^1), c = R_i(\Gamma_{ij}^1 \backslash \Gamma_{ij}^2)$ and $d = R_i(\Gamma_{ij}^2)$. If (2.7) holds, then when $d > 0$, we obtain from (2.6), $z \notin {}^{\delta^1}\Gamma_{ji}(\mathcal{A})$, $\Gamma_{ij}^1 \supseteq \Gamma_{ij}^2$, and Lemma 2.8.(I) that

$$\frac{|z - a_{i\cdots i}| - R_i(D_i \backslash \Gamma_{ij}^2)}{R_i(\Gamma_{ij}^2)} \frac{|z - a_{j\cdots j}| - R_j(D_j \backslash \Delta_{ij}^1)}{R_j(\Delta_{ij}^1)} \le 1,$$

which implies that $z \in {}^{\gamma^2}_{\delta^1}\Phi_{ij}(\mathcal{A})$. When $d = 0$, from (2.7), we easily obtain that

$$|z - a_{i\cdots i}| - R_i(D_i \backslash \Gamma_{ij}^2) \le R_i(\Gamma_{ij}^2) = 0.$$

Thus,

$$(|z - a_{i\cdots i}| - R_i(D_i \backslash \Gamma_{ij}^2))(|z - a_{j\cdots j}| - R_j(D_j \backslash \Delta_{ij}^1)) \le 0 \le R_i(\Gamma_{ij}^2)R_j(\Delta_{ij}^1),$$

which also implies $z \in {}^{\gamma^2}_{\delta^1}\Phi_{ij}(\mathcal{A})$. If (2.7) does not hold, namely,

$$\frac{|z - a_{i\cdots i}| - R_i(D_i \backslash \Gamma_{ij}^1)}{R_i(\Gamma_{ij}^1)} > 1,$$

then (2.8) holds. When $R_j(\Gamma_{ji}^2) > 0$, we obtain from (2.6), $\Delta_{ji}^1 \supseteq \Gamma_{ij}^1, \Delta_{ij}^1 \supseteq \Gamma_{ji}^2$, and Lemma 2.8 (I), (II) that

$$\frac{|z - a_{i\cdots i}| - R_i(D_i \backslash \Delta_{ji}^1)}{R_i(\Delta_{ji}^1)} \frac{|z - a_{j\cdots j}| - R_j(D_j \backslash \Gamma_{ji}^2)}{R_j(\Gamma_{ji}^2)} \le 1.$$

This means that $z \in {}^{\gamma^2}_{\delta^1}\Phi_{ji}(\mathcal{A})$. When $R_j(\Gamma_{ji}^2) = 0$, from (2.8), we easily obtain

$$|z - a_{j\cdots j}| - R_j(D_j \backslash \Gamma_{ji}^2) \le R_j(\Gamma_{ji}^2) = 0.$$

• If $z \in {}^{\delta^1}\Gamma_{ij}(\mathcal{A})$, then $z \in {}^{\gamma^2}_{\delta^1}\mathbf{\Phi}_{ji}(\mathcal{A})$.

- If $z \notin {}^{\delta^1}\Gamma_{ij}(\mathcal{A})$, then

$$(|z - a_{j\cdots j}| - R_j(D_j \backslash \Gamma_{ji}^2))(|z - a_{i\cdots i}| - R_i(D_i \backslash \Delta_{ji}^1)) \leq 0 \leq R_j(\Gamma_{ji}^2) R_i(\Delta_{ji}^1),$$

which also implies $z \in {}^{\gamma^2}_{\delta^1}\Phi_{ji}(\mathcal{A})$.

Therefore,

$${}^{\gamma^1}_{\delta^1}\mathbf{\Phi}_{ij}(\mathcal{A}) \subseteq {}^{\gamma^2}_{\delta^1}\mathbf{\Phi}_{ij}(\mathcal{A}) \cup {}^{\gamma^2}_{\delta^1}\mathbf{\Phi}_{ji}(\mathcal{A}).$$

Hence,

$${}^{\gamma^1}_{\delta^1}\mathbf{\Phi}(\mathcal{A}) \subseteq {}^{\gamma^2}_{\delta^1}\mathbf{\Phi}(\mathcal{A}).$$

Thus,

$${}^{\gamma^1}\Phi(\mathcal{A}) \subseteq {}^{\gamma^2}\Phi(\mathcal{A}).$$

From the above result, and the selection of γ_{ij}, δ_{ij} in the proofs of Corollaries 2.3, 2.5, 2.6, and 2.4, we obtain

$$\Phi(\mathcal{A}) \subseteq \Theta(\mathcal{A}) \subseteq \Omega(\mathcal{A}) \subseteq \Gamma(\mathcal{A}).$$

From [15, Theorem 2.2], we have

$$\Omega(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}).$$

From [10, Theorem 2.3], we get

$$\mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A}). \qquad \square$$

REMARK 2.10. From Proposition 2.9, we have

$$\Phi(\mathcal{A}) \subseteq {}^{\gamma}\Phi(\mathcal{A}), \qquad \forall \gamma_{ij} \subseteq N_i^{m-2}, \quad \forall (i,j) \in N \times N_i.$$

Based on Theorem 2.1, we can easily establish the following criterion to discern nonsingular tensors.

COROLLARY 2.11. *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{C}^{[m,n]}$ with $m, n \geq 2$. If for all $(i,j) \in N \times N_i$,*

$$(|a_{i\cdots i}| - R_i(D_i \backslash \Gamma_{ij}))(|a_{j\cdots j}| - R_j(D_j \backslash \Delta_{ij})) > R_i(\Gamma_{ij}) R_j(\Delta_{ij})$$

and

$$|a_{j\cdots j}| > R_j(D_j \backslash \Delta_{ij})$$

are fulfilled, then $\mathcal{A}$ is nonsingular, i.e., $0 \notin \sigma(\mathcal{A})$.

Proof. Let $\mathcal{A}$ be singular. Then $0 \in \sigma(\mathcal{A})$. From Theorem 2.1, we have

$$0 \in {}^{\gamma}_{\delta}\mathbf{\Phi}(\mathcal{A}) = {}^{\gamma}_{\delta}\Phi(\mathcal{A}) \cup {}^{\delta}\Gamma(\mathcal{A}).$$

Then there exists $(\mu_1, \mu_2) \in N \times N_{\mu_1}$ such that

$$\begin{aligned}(|a_{\mu_1\cdots\mu_1}| - R_{\mu_1}(D_{\mu_1} \backslash \Gamma_{\mu_1\mu_2}))\,(|a_{\mu_2\cdots\mu_2}| - R_{\mu_2}(D_{\mu_2} \backslash \Delta_{\mu_1\mu_2})) \\ \leq R_{\mu_1}(\Gamma_{\mu_1\mu_2}) R_{\mu_2}(\Delta_{\mu_1\mu_2})\end{aligned}$$

or

$$|a_{\mu_2\cdots\mu_2}| \le R_{\mu_2}(D_{\mu_2}\backslash\Delta_{\mu_1\mu_2}),$$

which is a contradiction. So, $\mathcal{A}$ is nonsingular. The proof is completed. ∎

REMARK 2.12. The tensor in Corollary 2.11 is called a ${}^{\gamma}_{\delta}\mathbf{\Phi}$-tensor or a (γ,δ)-doubly strictly diagonally dominant ((γ,δ)-DSDD) tensor, and the conditions are called ${}^{\gamma}_{\delta}\mathbf{\Phi}$-conditions or (γ,δ)-doubly strictly diagonally dominant ((γ,δ)-DSDD) conditions. The nonstrict conditions are called ${}^{\gamma}_{\delta}\mathbf{\Phi}_0$-conditions or (γ,δ)-doubly diagonally dominant ((γ,δ)-DDD) conditions, and the tensor which satisfy the nonstrict conditions is called a ${}^{\gamma}_{\delta}\mathbf{\Phi}_0$-tensor or a (γ,δ)-doubly diagonally dominant ((γ,δ)-DDD) tensor.

3. Numerical examples. In this section, in order to demonstrate the superiority of Theorem 2.1 and Proposition 2.9, in particular $\Theta(\mathcal{A})$ in Corollary 2.5 and $\Phi(\mathcal{A})$ in Corollary 2.3, we present two numerical examples.

EXAMPLE 3.1. Consider the tensor $\mathcal{A}_1 = (a_{i_1i_2i_3i_4}) \in \mathbb{C}^{[4,4]}$ with

$$a_{1444} = 8,\ a_{2333} = \mathbf{i},\ a_{3222} = 2,\ a_{3333} = 4\mathbf{i},\ a_{4111} = 1,\ a_{1233} = 3,\ a_{1234} = 4,$$

and all other entries $a_{i_1i_2i_3i_4} = 0$. By using Mathematica, the set $\Theta(\mathcal{A}_1)$ and the set $\Omega(\mathcal{A}_1)$ are displayed in Figure 3.1. By noting that $(\pm 5, -5) \in \Omega(\mathcal{A}_1)\backslash\Theta(\mathcal{A}_1)$, we know that $\Theta(\mathcal{A}_1)$ is a proper subset of $\Omega(\mathcal{A}_1)$. Thus, $\Theta(\mathcal{A}_1)$ provides more precise information about the location of the eigenvalues of $\mathcal{A}_1$ than $\Omega(\mathcal{A}_1)$ does.

EXAMPLE 3.2. Consider the tensor $\mathcal{A}_2 = (a_{i_1i_2i_3i_4}) \in \mathbb{C}^{[4,4]}$ with

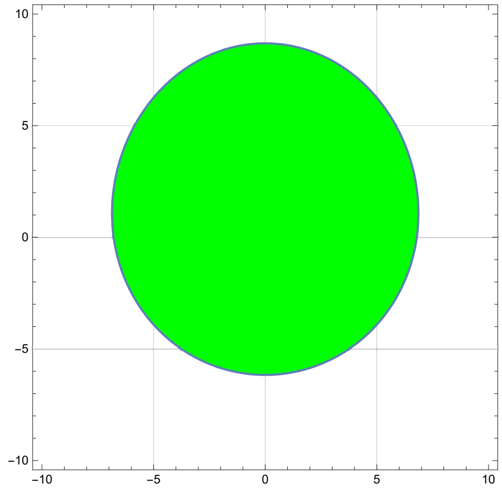
$$a_{1111} = \mathbf{i},\ a_{1222} = 1,\ a_{1444} = 8,\ a_{2333} = \mathbf{i},\ a_{3333} = 2\mathbf{i},\ a_{4333} = 1,\ a_{4444} = 4,$$
$$a_{1233} = 2,\ a_{1234} = 3,$$

and $a_{i_1i_2i_3i_4} = 0$ otherwise. By using Mathematica, the set $\Phi(\mathcal{A}_2)$ and the set $\Theta(\mathcal{A}_2)$ are displayed in Figure 3.2. By noting that $(0, 5) \in \Theta(\mathcal{A}_2)\backslash\Phi(\mathcal{A}_2)$, we know that $\Phi(\mathcal{A}_2)$ is a proper subset of $\Theta(\mathcal{A}_2)$. Thus, $\Phi(\mathcal{A}_2)$ provides a tighter bound for the eigenvalues of $\mathcal{A}_2$ than $\Theta(\mathcal{A}_2)$ does.

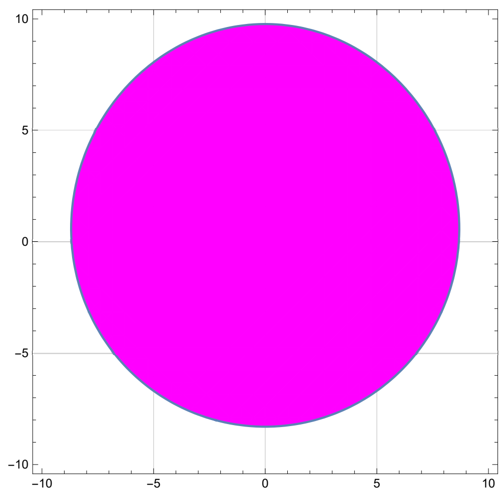
Acknowledgments. The authors would like to thank the editors and the anonymous referees for their valuable suggestions, comments and remarks, which have considerably improved the presentation of the paper.

REFERENCES

[1] A. BRAUER, *Limits for the characteristic roots of a matrix II*, Duke Math. J., 14 (1947), pp. 21–26.
[2] C. BU, Y. WEI, L. SUN, AND J. ZHOU, *Brualdi-type eigenvalue inclusion sets of tensors*, Linear Algebra Appl., 480 (2015), pp. 168–175.
[3] S. GERŠGORIN, *Über die Abgrenzung der Eigenwerte einer Matrix*, Izv. Akad. Nauk SSSR Ser. Mat., 1 (1931), pp. 749–754.
[4] C. J. HILLAR, AND L. H. LIM, *Most tensor problems are NP-hard*, J. ACM, 60 (2013), Art. 45, 39 pages.
[5] S. L. HU, Z. H. HUANG, C. LING, AND L. QI, *On determinants and eigenvalue theory of tensors*, J. Symbolic Comput., 50 (2013), pp. 508–531.
[6] Z. HUANG, L. WANG, Z. XU, AND J. CUI, *A new S-type eigenvalue inclusion set for tensors and its applications*, J. Inequal. Appl, 2016 (2016) Paper No. 254, 19 pages.
[7] C. Q. LI, Z. CHEN, AND Y. T. LI, *A new eigenvalue inclusion set for tensors and its applications*, Linear Algebra Appl., 481 (2015), pp. 36–53.
[8] C. Q. LI, A. Q. JIAO, AND Y. T. LI, *An S-type eigenvalue localization set for tensors*, Linear Algebra Appl., 493 (2016), pp. 469–483.
[9] C. Q. LI AND Y. T. LI, *An eigenvalue localization set for tensors with applications to determine the positive (semi-)definiteness of tensors*, Linear Multilinear Algebra, 64 (2016), pp. 587–601.



(a) The set $\Theta(\mathcal{A}_1)$.



(b) The set $\Omega(\mathcal{A}_1)$.

FIG. 3.1. *Comparison between* $\Theta(\mathcal{A}_1)$ *and* $\Omega(\mathcal{A}_1)$ *for* $\mathcal{A}_1$ *specified in Example 3.1.*

[10] C. Q. LI, Y. T. LI, AND X. KONG, *New eigenvalue inclusion sets for tensors*, Numer Linear Algebra Appl., 21 (2014), pp. 39–50.

[11] C. Q. LI, J. J. ZHOU, Y. T. LI, *A new Brauer-type eigenvalue localization set for tensors*, Linear Multilinear Algebra, 64 (2016), pp. 727–736.

[12] L. H. LIM, *Singular values and eigenvalues of tensors: a variational approach*, in: CAMSAP'05: Proceeding of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Proceeding, 2005, IEEE, Piscataway, 2005, pp. 129–132.

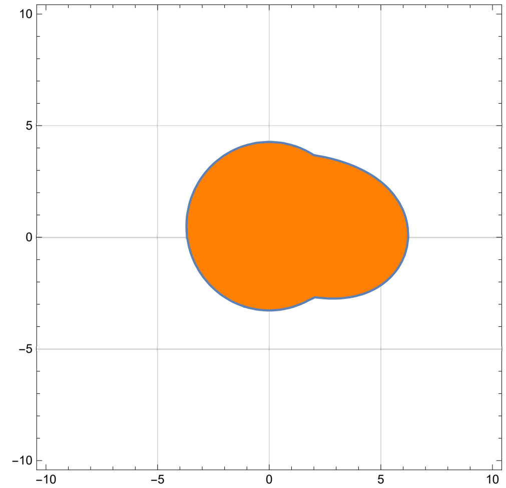
[13] L. QI, *Eigenvalues of a supersymmetric tensor and positive definiteness of an even degree multivariate form*, Tech. Rep., Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, 2004.

[14] ———, *Eigenvalues of a real supersymmetric tensor*, J. Symbolic. Comput., 40 (2005), pp. 1302–1324.

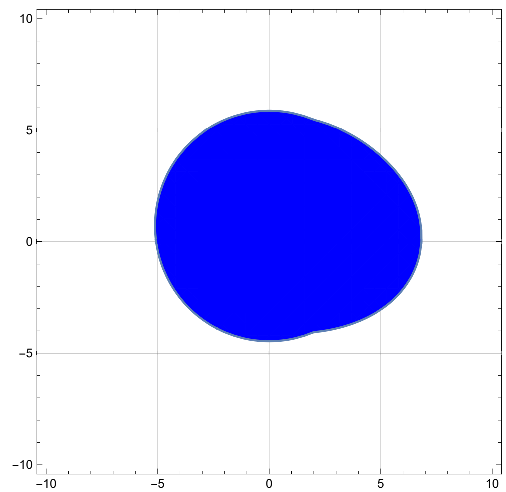
[15] C. L. SANG AND J. X. ZHAO, *A new eigenvalue inclusion set for tensors with its applications*, Cogent Math., 4 (2017), Art. Id. 1320831, 13 pages.

[16] R. S. VARGA, *Geršgorin and His Circles*, Springer, Berlin, 2004.

[17] X. Z. WANG, Y. M. WEI, *$\mathscr{H}$-tensors and nonsingular $\mathscr{H}$-tensors*, Front. Math. China, 11 (2016), pp. 555–575.



(a) The set $\Phi(\mathcal{A}_2)$.



(b) The set $\Theta(\mathcal{A}_2)$.

FIG. 3.2. *Comparison between $\Phi(\mathcal{A}_2)$ and $\Theta(\mathcal{A}_2)$ for $\mathcal{A}_2$ specified in Example 3.2.*

[18] Y. YANG, Q. YANG, *Further results for Perron-Frobenius theorem for nonnegative tensors*, SIAM J. Matrix Anal. Appl., 31 (2010), pp. 2517–2530.
[19] Q. YANG, Y. YANG, *Further results for Perron-Frobenius theorem for nonnegative tensors II*, SIAM J. Matrix Anal. Appl., 32 (2011), pp. 1236–1250.