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On Additive Functions Fulfilling some Additional Condition*

Von

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Abstract

Let $D \subset \mathbb{R}^2$ be an arbitrary set. We consider the following question: What kind of assumptions on D imply that every additive function $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$(x, y) \in D \Rightarrow f(x)f(y) = 0 \qquad (1)$$

is identically equal to zero? It is true if D is a non-empty open subset of $\mathbb{R}^2$. G. Szabó posed this problem for $D = \{(x, y); x^2 + y^2 = 1\}$ ([7]). We give an affirmative answer to Szabó's question and, moreover, we give some sufficient conditions to obtain the above assertion in much more general spaces.

1.

Let X and Y be linear spaces over the rationals $\mathbb{Q}$. A function $f : X \to Y$ is called additive if it satisfies the (Cauchy's) equation

$$f(x+y) = f(x) + f(y), \quad x, y \in X. \qquad (2)$$

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Every additive function is uniquely determined by its values on a so-called Hamel base (i.e. a base of X over the rationals) and it fulfills the condition

$$f(rx) = rf(x), \quad x \in X, \quad r \in \mathbb{Q},$$

(cf. [4], for example). We start our considerations with two examples.

Example 1. Let $f : \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ is here and in the sequel the set off all reals) be a discontinuous additive function vanishing on a saturated non-measurable in the Lebesgue sense subset S ([4] p. 58 and 297-Th. 7) and put $D = (S \times \mathbb{R}) \cup (\mathbb{R} \times S)$. Evidently f is not identically equal to zero and condition (1) is fulfilled.

The set D, though large in a sense, does not contain any segment. The following second example shows that even when D contains a segment it is possible to find a non-zero additive function fulfilling condition (1).

Example 2. Let $f : \mathbb{R}$ be a discontinuous additive function such that the restriction of f to the set of all rationals is equal to zero. If $D = \{(x, y); \max\{|x|, |y|\} = 1\}$, then f satisfies condition (1) and is not identically equal to zero.

The set from Example 2 is a unit circle if we treat $\mathbb{R}^2$ as a linear space endowed with the norm $\|(x, y)\| := \max\{|x|, |y|\}$. We shall show that the answer to our (Szabó's) question is positive if we take a different norm in $\mathbb{R}^2$. We have the following

Remark 1. Let $D = \{(x, y) \in \mathbb{R}^2; |x| + |y| = 1\}$ and let $f : \mathbb{R} \to \mathbb{R}$ be an additive function fulfilling condition (1). Then f is identically equal to zero.

Proof: According to our assumptions we obtain

$$f(x)f(1 - x) = 0, \quad x \in (0, 1). \tag{3}$$

If $f(1) = 0$ it follows from (3) that $f(x) = 0, x \in (0, 1)$, and hence $f \equiv 0$. Assume

$$f(1) \neq 0 \tag{4}$$

and take an $x_0 \in (0, 1)$ such that $f(x_0) = 0$. For arbitrary $z \in \mathbb{R}$ there exists an integer n such that $z + nx_0 \in (0, 1)$. Thus

$$f(z + nx_0)f(1 - z - nx_0) = 0,$$

which, because of the additivity of f, yields the condition

$$f(z)f(1 - z) = 0, \quad z \in \mathbb{R}.$$

Putting $-z$ instead of z in this equality and adding both we get

$$f(z)f(1) = 0,$$

which contradicts (4) and proves Remark 1.

A positive answer to Szabó's question is contained in

Theorem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function fulfilling condition (1) where $D = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$. Then f is identically equal to zero.

Proof: Take an arbitrary $x \in (0, 1)$ and choose a y such that $x^2 + y^2 = 1$. Setting

$$u = \frac{3x + 4y}{5}, \quad v = \frac{4x - 3y}{5}$$

we observe that

$$u^2 + v^2 = x^2 + y^2 = 1.$$

By virtue of (1)

$$f(u)f(v) = f(x)f(y) = 0. \tag{5}$$

Moreover, by (5)

$$\begin{aligned} 0 = f(u)f(v) &= \frac{1}{25}[3f(x) + 4f(y)][4f(x) - 3f(y)] \\ &= \frac{12}{25}(f(x)^2 - f(y)^2) \end{aligned}$$

and hence $f(x)^2 = f(y)^2$. On account of (5) $f(x) = 0$. Due to the arbitraryness of $x (\in [0, 1))$, f is identically equal to zero because it is additive.

Corollary 1. A similar result holds true if $D = \{(x,y) \in \mathbb{R}^2; x^2 + y^2 = r^2\}$, where $r > 0$ is an arbitrary constant.

Proof: The function $F(x) = f(rx), x \in \mathbb{R}$ fulfills all assumptions of Theorem 1. We have also

Theorem 2. Let X be a real normed space and let Y be an arbitrary linear space. If $f : X \to Y$ is an arbitrary additive function fulfilling the condition

$$\|x\|^2 + \|y\|^2 = 1 \Rightarrow f(x) = 0 \text{ or } f(y) = 0,$$

then f is identically equal to zero.

Proof: First let us assume that $\dim X > 1$. Take an arbitrary $x \in X$ such that $\|x\| = \frac{\sqrt{2}}{2}$ and put $y = x$. Then

$$\|x\|^2 + \|y\|^2 = 1$$

and by our assumption we get

$$f(x) = 0,$$

which means that f vanishes on a circle $C = \{x \in X : \|x\| = \frac{\sqrt{2}}{2}\}$. Since $\dim X \geq 2$ for every $u \in X$, $\|u\| < \frac{\sqrt{2}}{2}$, there exist $v_1, v_2 \in C$ such that $v_1 + v_2 = u$ ([1], see the proof of Lemma 1). Consequently

$$f(u) = f(v_1 + v_2) = f(v_1) + f(v_2) = 0.$$

Thus f, being an additive function vanishing on a ball, has to be identically equal to zero.

If $\dim X = 0$, the assertion is trivial. If, finally, $\dim X = 1$ we may assume that $X = \mathbb{R}$ and that $\|x\| = r^{-1}|x|$ for some $r > 0$. Thus for every linear functional $\varphi : Y \to \mathbb{R}$ the function $\varphi \circ f : X \to \mathbb{R}$ satisfies the assumptions of Corollary 1, implying that $\varphi \circ f = 0$. But the linear functionals on Y separate the points of Y. Thus $f = 0$.

Let G be an abelian group and let $\mathbb{K}$ be a field of characteristic zero. For mappings $w : G \to \mathbb{K}$ and an element $h \in G$ the difference operator Δ_h is defined by

$$\Delta_h w(x) := w(x + h) - w(x).$$

A mapping $w : G \to \mathbb{K}$ is called a generalized polynomial of degree less than $n + 1$ iff

$$\Delta_h^{n+1} w(x) = 0, \quad x, h \in G,$$

where Δ^k denotes the $k - th$ iterate of Δ.

Theorem 3. Let $f : G \to \mathbb{K}$ be an additive function and let

$$D = \{(v(x), w(x)) \in \mathbb{K} \times \mathbb{K}; x \in G)\},$$

where $v, w : G \to \mathbb{K}$ are generalized polynomials such that $\text{lin}_{\mathbb{Q}}\, v(G) = \text{lin}_{\mathbb{Q}}\, w(G) = \mathbb{K}$. If f fulfills condition (1), then it is identically equal to zero.

Proof: By our assumptions

$$f(v(x))f(w(x)) = 0, \quad x \in G. \tag{6}$$

Since $f \circ v$ and $f \circ w$ are generalized polynomials we can apply a result of F. Halter-Koch, L. Reich and J. Schwaiger ([3], Th. 2). Therefore

$f \circ v \equiv 0$ or $f \circ w \equiv 0$. It follows from the equality $\mathrm{lin}_{\mathbb{Q}}\, v(G) = \mathrm{lin}_{\mathbb{Q}}\, w(G) = \mathbb{K}$ that f is identically equal to zero.

Remark 2. The assumption $\mathrm{lin}_{\mathbb{Q}}\, v(G) = \mathrm{lin}_{\mathbb{Q}}\, w(G) = \mathbb{K}$ is essential in Theorem 3.

This can be seen by taking $v = \mathrm{id}$ and $w = f$, where f is a function as defined in Example 2.

Corollary 2. Let $v, w : \mathbb{R} \to \mathbb{R}$ be arbitrary (ordinary) polynomials of degree at least one. If $f : \mathbb{R} \to \mathbb{R}$ is an additive function fulfilling condition (6) then it is identically equal to zero.

This is so, since $v(\mathbb{R})$ and $w(\mathbb{R})$ are non-trivial intervals.

Condition (1) may be generalized by replacing the righthand side of the implication (i.e. $f(x)f(y) = 0$ for $(x, y) \in D$) by $Q(f(x), f(y)) = 0$ for all $(x, y) \in D$, where Q is a polynomial in indeterminates X and Y over $\mathbb{R}(Q \in \mathbb{R}[X, Y])$. This means that we now are interested in conditions on $D \subseteq \mathbb{R}^2$ such that

$$(x, y) \in D \Rightarrow Q(f(x), f(y)) = 0 \tag{1'}$$

for an additive function $f : \mathbb{R} \to \mathbb{R}$ implies $f = 0$.

In this situation we will show

Theorem 3′. Let $f : \mathbb{R} \to \mathbb{R}$ be additive and let p and q be generalized polynomials of degree 1, i.e. $p = g + a, q = h + b$, where $g, h : \mathbb{R} \to \mathbb{R}$ are additive and *a, b* real constants. Assume that $p(\mathbb{R})$ and $q(\mathbb{R})$ contain Hamel bases. Furthermore, let $Q \in \mathbb{R}[X, Y]$ such that no polynomial $AX + BY + C$ with $AB \neq 0$ divides $Q(X, Y)$, and let

$$D := \{(p(u), q(u)) | u \in \mathbb{R}\} \subseteq \mathbb{R}^2.$$

Then, if

$$(x, y) \in D \Rightarrow Q(f(x), f(y)) = 0, \tag{1'}$$

we have $f = 0$.

Proof: We have $f(p(u)) = (f \circ g)(u) + c, f(q(u)) = (f \circ h)(u) + d$, where $c = f(a), d = f(b). f \circ g$ and $f \circ h$ are additive, and since $p(\mathbb{R}), q(\mathbb{R})$ contain Hamel bases, the same holds for $g(\mathbb{R}), h(\mathbb{R})$, and so $f \circ g \neq 0, f \circ h \neq 0$. By (1′) we have

$$Q((f \circ g)(u) + c, (f \circ h)(u) + d) = 0, \quad u \in \mathbb{R}.$$

We denote by $Q_1(X, Y)$ the polynomial $Q_1(X, Y) := Q(X + c, Y + d)$, where $Q_1 \not\equiv 0, Q_1((f \circ g)(u), (f \circ h)(u)) = 0, u \in \mathbb{R}$.

By [6, theorem 1] we get that $f \circ g, f \circ h$ are linearly dependent over $\mathbb{R}$, i.e. there exists $(\lambda, \mu) \in \mathbb{R}^2, (\lambda, \mu) \neq (0,0)$ such that

$$\lambda(f \circ g) + \mu(f \circ h) = 0. \tag{7}$$

Since $f \circ g \neq 0, f \circ h \neq 0$ we deduce that $\lambda \neq 0, \mu \neq 0$. But then by [6, theorem 2] we see that

$$\lambda X + \mu Y | Q_1(X, Y),$$

and therefore

$$\lambda X + \mu Y - (\lambda c + \mu d) | Q(X, Y),$$

where $\lambda\mu \neq 0$, which contradicts the assumption of the theorem. So we have necessarily $f = 0$, which concludes the proof.

The set D from Example 1 is large in a certain sense; it is saturated non-measurable in the Lebesgue sense as well as it is a second category set without Baire property. However, we prove the following

Theorem 4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be an additive function fulfilling condition (1) and assume that $D \subset \mathbb{R}^{2n}$ is a Lebesgue measurable subset with positive measure. Then f is identically equal to zero.

Proof: The set

$$H := \{x \in \mathbb{R}^n; f(x) = 0\}$$

is a subgroup of $\mathbb{R}^n$ and since $D \subset (H \times \mathbb{R}^n) \cup (\mathbb{R}^n \times H)$ the outer Lebesgue measure of H is positive. It is not hard to check hat H is dense in $\mathbb{R}^n$. By Smítal's lemma ([4], [5]) the set $G := (H \times H) + D$ is of full Lebesgue measure in $\mathbb{R}^{2n}$ (in fact; since $\mathbb{R}^n$ is separable there exists a countable subset H_0 of H which is dense in $\mathbb{R}^n$, and by Smítal's lemma the set $(H_0 \times H_0) + D$ has full Lebesgue measure in $\mathbb{R}^{2n}$ and, of course, $(H_0 \times H_0) + D \subset G$). Moreover, for every $(x,y) \in G$ we have $x = h_1 + d_1, y = h_2 + d_2, h_1, h_2 \in H, (d_1, d_2) \in D$, and hence $f(x)f(y) = f(d_1) f(d_2) = 0$. Therefore

$$G \subset (H \times \mathbb{R}^n) \cup (\mathbb{R}^n \times H) =: S.$$

We will show that H is measurable in the Lebesgue sense and of the full measure in $\mathbb{R}^{2n}$. By Fubini's theorem the set

$$B := \{x \in \mathbb{R}^n; S_x = \{y \in \mathbb{R}^n; (x,y) \in S\} \text{ is measurable}\}$$

is measurable in the Lebesgue sense and of full measure in $\mathbb{R}^n$. If $B \subset H$, then H is measurable and of the full measure in $\mathbb{R}^n$. If $B \backslash H \neq \emptyset$, take an $x \in B \backslash H$. Then $S_x = H$ and $x \in B$. So, H is measurable, too. Thus H, being a dense subgroup of full measure in $\mathbb{R}^n$, is equal to $\mathbb{R}^n$. (In fact, any subgroup of $\mathbb{R}^n$ of positive measure equals $\mathbb{R}^n$: Assume that H is a full measure group in $\mathbb{R}^n$. Take an arbitrary x from $\mathbb{R}^n$. Then the set $x - H$ is also full measure in $\mathbb{R}^n$ and therefore by the Steinhaus theorem the intersection $H \cap (x - H)$ is a nonempty set. Choosing a z from this intersection we get that $x = z + (x - z)$ belongs to $H + H = H$. Thus $H = \mathbb{R}^n$.)

The proof of Theorem 4 is finished.

A topological analogue of Theorem 4 is also true. One can prove the following

Theorem 5. Let D be a second category subset of $\mathbb{R}^{2n}$ with the Baire property and let $f : \mathbb{R}^n \to \mathbb{R}$ be an additive function fulfilling condition (1). Then f is identically equal to zero.

Proof: The proof is quite similar to the proof of Theorem 4 because Fubini's theorem and Smítal's lemma have topological analogues ([2], [4]).

The results of Remark 1 and Theorem 1 can be viewed as special cases of the following.

Theorem 6. Let $u, v : T \to \mathbb{R}$ be such that for all $t \in T$ there is some $t_1 \in T$ and some 2×2-matrix Q with rational and nonvanishing entries a, b, c, d such that $(u(t_1), v(t_1))^T = Q(u(t), v(t))^T$. Moreover let $u(T)$ or $v(T)$ generate $\mathbb{R}$ as a $\mathbb{Q}$-vector space. Then we have that the condition

$$(f \circ u) \cdot (f \circ v) = 0$$

implies $f = 0$.

Proof: Fix $t \in T$. Without loss of generality we may suppose that $f(u(t)) = 0$. Choosing t_1 and Q as above and using the fact that $f(u(t_1)) \cdot f(v(t_1)) = 0$ we get

$$\begin{aligned} 0 &= f(au(t) + bv(t)) f(cu(t) + dv(t)) \\ &= acf(u(t))^2 + adf(u(t))f(v(t)) + bcf(v(t))f(u(t)) + bdf(v(t))^2 \\ &= bdf(v(t))^2, \end{aligned}$$

implying that $f(v(t)) = 0$. Thus $f \circ u = f \circ v = 0$ which gives us the desired result.

Remark 3. Using $u =$ cos and $v =$ sin we get Theorem 1 with $t_1 = t + t_0$ where t_0 is such that $\cos(t_0) = 3/5$ and $\sin(t_0) = 4/5$, for example. Remark 1 may be considered as the case $T =]0, 1[$, $u(t) = t$, $v(t) = 1 - t$, $a = b = c = d = 1/2$.

A different example (hyperbola) is given by $u = \cosh$, $v = \sinh$, $t_1 = t + t_0$, where now t_0 is choosen in such a way that both $\cosh(t_0)$ and $\sinh(t_0)$ are positive rationals (which of course is possible).

References

[1] Ger, R.: Some remarks on quadratic functionals. Glasnik Mat. **23**, 315–330 (1988).

[2] Ger, R., Kominek, Z. K., Sablik, M.: Generalized Smítal's Lemma and a Theorem of Steinhaus. Radovi Mat. **1**, 101–119 (1985).

[3] Halter-Koch, F., Reich, L., Schwaiger, J.: On products of additive functions. Aequationes Math. **45**, 83–88 (1993).

[4] Kuczma, M.: An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality. Polish Scientific Publishers and Silesian University Press, Warszawa-Kraków-Katowice, 1985.

[5] Kuczma, M., Smítal, J.: On measures connected with the Cauchy equation. Aequationes Math. **14**, 421–428 (1976).

[6] Reich, L., Schwaiger, J.: On polynomials in additive and multiplicative functions. In: J. Aczél (ed.) Functional Equations: History, Applications and Theory. D. Reidel Publishing Company, Dordrecht/Boston/Lancaster, 1984, 127–160.

[7] Szabó, G.: Report of Meeting The Thirtieth International Symposium on Functional Equations, Oberwolfach, Germany 1992. Aequationes Math. **46**, p. 294 (1993).

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