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A Note on Orthopseudorings and Boolean Quasirings*

By

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Abstract

The concept of an *orthopseudoring* was introduced in [2], where it was shown that there is a one-to-one correspondence between orthopseudorings and ortholattices, generalizing the well-known correspondence between Boolean rings and Boolean algebras (see Prop. 1.1 below).

Independently, in [6] other ring-like structures, so called *Boolean quasirings* were defined, and again a one-to-one correspondence between these structures and ortholattices was established (see Prop. 1.2).

In this paper, we first study connections between orthopseudorings and Boolean quasirings, which are both based on generalizations of symmetric differences (see e.g. [4]). Then we introduce the concept of an ideal and classify those ideals which are congruence kernels. We show that, contrary to rings, an ideal can be the congruence kernel of more than one congruence.

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1. Preliminaries

Definition 1.1. An algebra $\mathbf{L} = (L; \vee, \wedge, 0, 1, ^{\perp})$ of type (2, 2, 0, 0, 1) is called a *bounded lattice with an involutory antiautomorphism* (BLIA) if $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and the following laws hold:

(a) $(x^{\perp})^{\perp} = x$
(b) $(x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp}$
(c) $(x \wedge y)^{\perp} = x^{\perp} \vee y^{\perp}$

A BLIA $\mathbf{L}$ is called an *ortholattice* if the laws

(d) $x \vee x^{\perp} = 1$ and
(e) $x \wedge x^{\perp} = 0$

hold in $\mathbf{L}$. An ortholattice is called an *orthomodular lattice* if it satisfies

(f) $(x \wedge y) \vee ((x \wedge y)^{\perp} \wedge y) = y$.

Basic facts on ortholattices and orthomodular lattices can be found in [1] and [9].

In the following, we recall some concepts and results of [2] and [6].

Definition 1.2. An algebra $\mathbf{P} = (P; +, \cdot, 0, 1)$ of type (2, 2, 0, 0) is called an *orthopseudoring* (OPR) if the following laws hold:

$$x + y = y + x \tag{1}$$

$$x + 0 = x \tag{2}$$

$$x(yz) = (xy)z \tag{3}$$

$$xy = yx \tag{4}$$

$$xx = x \tag{5}$$

$$x \cdot 0 = 0 \tag{6}$$

$$x \cdot 1 = x \tag{7}$$

$$1 + (1 + xy)(1 + y) = y \tag{8}$$

$$(1 + x(1 + y))(1 + y(1 + x)) = 1 + (x + y) \tag{9}$$

$$x + x = 0 \tag{10}$$

$$x + (1 + y) = (x + 1) + y \tag{11}$$

$$x(1 + xy) = x + xy \tag{12}$$

Remark 1.1. In fact, in [2] the law

$$(1 + xy)(1 + y) = 1 + y \tag{8a}$$

is used instead of (8), but it is easy to see that both definitions are equivalent: By adding 1 to both sides and using the equations

$$1 + (1 + z) = (1 + 1) + z = 0 + z = z,$$

we see that (8) follows from (8a) and conversely.

The following result was proved in [2]:

Proposition 1.1.
(i) If $\mathbf{L} = (L; \vee, \wedge, 0, 1, ^{\perp})$ is an ortholattice, $x + y := (x \wedge y^{\perp}) \vee (x^{\perp} \wedge y)$ and $x \cdot y := x \wedge y$, then $\mathbf{P}(\mathbf{L}) := (L; +, \cdot, 0, 1)$ is an OPR.
(ii) If $\mathbf{P} = (P; +, \cdot, 0, 1)$ is an OPR, $x \vee y := 1 + (1 + x)(1 + y)$, $x \wedge y := x \cdot y$ and $x^{\perp} := 1 + x$, then $\mathbf{L}(\mathbf{P}) := (P; \vee, \wedge, 0, 1, ^{\perp})$ is an ortholattice.
(iii) $\mathbf{L}(\mathbf{P}(\mathbf{L})) = \mathbf{L}$ and $\mathbf{P}(\mathbf{L}(\mathbf{P})) = \mathbf{P}$.

Definition 1.3. An Algebra $\mathbf{R} = (R; +, \cdot, 0, 1)$ of type (2, 2, 0, 0) is called a *generalized Boolean quasiring* (GBQR) if it satisfies (1)–(8) of Definition 1.2. A GBQR is called a *Boolean quasiring* (BQR) if it satisfies (10) and

$$(1 + (1 + x)(1 + y))(1 + xy) = x + y. \tag{9*}$$

The following result was proved in [6]:

Proposition 1.2.
(i) If $\mathbf{L} = (L; \vee, \wedge, 0, 1, ^{\perp})$ is a BLIA, $x + y := (x \vee y) \wedge (x \wedge y)^{\perp}$ and $x \cdot y := x \wedge y$, then $\mathbf{R}(\mathbf{L}) := (L; +, \cdot, 0, 1)$ is a GBQR with (9*).
(ii) If $\mathbf{R} = (R; +, \cdot, 0, 1)$ is a GBQR, $x \vee y := 1 + (1 + x)(1 + y)$, $x \wedge y := x \cdot y$ and $x^{\perp} := 1 + x$, then $\mathbf{L}(\mathbf{R}) := (R; \vee, \wedge, 0, 1, ^{\perp})$ is a BLIA.
(iii) If $\mathbf{L}$ is a BLIA and $\mathbf{R}$ a GBQR with (9*), then $\mathbf{L}(\mathbf{R}(\mathbf{L})) = \mathbf{L}$ and $\mathbf{R}(\mathbf{L}(\mathbf{R})) = \mathbf{R}$.
(iv) $\mathbf{L}$ is an ortholattice if and only if $\mathbf{R}(\mathbf{L})$ is a BQR.

Remark 1.2. If $\mathbf{L}$ is an orthomodular lattice, then $\mathbf{P}(\mathbf{L}) = \mathbf{R}(\mathbf{L})$ if and only if $\mathbf{L}$ is a Boolean algebra (see e.g. [4], Theorem 3.7). So if $\mathbf{L}$ is any orthomodular lattice which is not Boolean, then $\mathbf{P}(\mathbf{L}) \neq \mathbf{R}(\mathbf{L})$, and hence $\mathbf{P}(\mathbf{L})$ is *not* a BQR: If $\mathbf{P}(\mathbf{L})$ would be a BQR, then by Prop. 1.1 and 1.2 we would have $\mathbf{P}(\mathbf{L}) = \mathbf{R}(\mathbf{L}(\mathbf{P}(\mathbf{L}))) = \mathbf{R}(\mathbf{L})$, a contradiction. Similarly, in this case $\mathbf{R}(\mathbf{L})$ is *not* an OPR.

Properties of GBQR's which are of interest in axiomatic quantum mechanics are studied in [5] and [7].

2. The Correspondence between OPR's and BQR's

Lemma 2.1. *Every GBQR* satisfies the equation

$$1 + (1 + x) = x. \tag{13}$$

Proof: Put $x = y$ in (8) and use (5): $x = 1 + (1 + xx)(1 + x) = 1 + (1 + x)$. □

Lemma 2.2.

(i) Every GBQR with (9) satisfies (11).
(ii) Every GBQR with (9) satisfies (11).*

Proof:

(i) If we put $1 + x$ instead of x in (9) and use (13), we obtain: $(1 + (1 + x)(1 + y))(1 + yx) = 1 + ((1 + x) + y)$, and for $1 + y$ instead of y we obtain: $(1 + xy)(1 + (1 + y)(1 + x)) = 1 + (x + (1 + y))$. By (4), the expressions on the left sides are equal, hence $1 + ((1 + x) + y) = 1 + (x + (1 + y))$. By using (13) and (1) we get (11).
(ii) The proof is similar to that of (i). □

Lemma 2.3.

(i) Every GBQR with (9) and (10) satisfies (12) and

$$x(1 + x) = 0. \tag{14}$$

(ii) Every BQR satisfies (12) and (14).

Proof:

(i) By (5), (9) (with $x = y$) and (10) we have: $1 + x(1 + x) = (1 + x(1 + x))(1 + x(1 + x)) = 1 + (x + x) = 1 + 0$, hence by (13) $x(1 + x) = 0$. In (9) put xy instead of y, then $(1 + x(1 + xy))(1 + (xy)(1 + x)) = 1 + (x + xy)$. By (3), (4) and (14) we have $(xy)(1 + x) = 0$, hence by (2) and (7) $1 + x(1 + xy) = 1 + (x + xy)$, and by using (13) we get (12).
(ii) By (13), (5), (9*) (with $x = y$) and (10) we have: $x(1 + x) = (1 + (1 + x))(1 + x) = (1 + (1 + x)(1 + x))(1 + xx) = x + x = 0$. In (9*) put xy instead of y, then $(1 + (1 + x)(1 + xy))(1 + x(xy)) = x + xy$. By (3) and (5) $x(xy) = xy$, and by (4) and (8) $1 + (1 + x)(1 + xy) = x$, which yields (12). □

From Lemma 2.1–2.3 we get:

Theorem 2.1.

(i) **P** *is an OPR if and only if* **P** *is a GBQR with (9) and (10).*
(ii) Every BQR satisfies (11) and (12).

If $\mathbf{L} = (L; \vee, \wedge, 0, 1, ^{\perp})$ is an ortholattice, $\mathbf{P}(\mathbf{L}) = (L; +, \cdot, 0, 1)$ and $\mathbf{R}(\mathbf{L}) = (L; \oplus, \cdot, 0, 1)$, then we have $x \oplus y = (x \vee y) \wedge (x \wedge y)^{\perp} = ((x \wedge (y^{\perp})^{\perp}) \vee (x^{\perp} \wedge y^{\perp}))^{\perp} = (x + y^{\perp})^{\perp}$, i.e. $x \oplus y = 1 + (x + (1 + y))$. In the following we give a generalization of this fact.

We define, for any algebra $\mathbf{P} = (P; +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$, $\mathbf{Q}(\mathbf{P}) := (P; \oplus, \cdot, 0, 1)$ where $x \oplus y := 1 + (x + (1 + y))$, and $\mathbf{L}(\mathbf{P}) := (L; \vee, \wedge, 0, 1, ^{\perp})$ where $x \vee y := 1 + (1 + x)(1 + y)$, $x \wedge y := x \cdot y$ and $x^{\perp} := 1 + x$.

Theorem 2.2.

(i) *If $\mathbf{P}$ is a GBQR, then $\mathbf{Q}(\mathbf{Q}(\mathbf{P})) = \mathbf{P}$ and $\mathbf{L}(\mathbf{Q}(\mathbf{P})) = \mathbf{L}(\mathbf{P})$.*

(ii) *$\mathbf{P}$ is a GBQR with (11) if and only if $\mathbf{Q}(\mathbf{P})$ is a GBQR with (11).*

(iii) *$\mathbf{P}$ is a GBQR with (9) if and only if $\mathbf{Q}(\mathbf{P})$ is a GBQR with (9*).*

(iv) *$\mathbf{P}$ is an OPR if and only if $\mathbf{Q}(\mathbf{P})$ is a BQR.*

(v) *$\mathbf{P}$ is an OPR with*

$$(x + xy) + xy = x \tag{15}$$

if and only if $\mathbf{Q}(\mathbf{P})$ is a BQR with

$$(1 \oplus xy) \oplus (x \oplus xy) = 1 \oplus x. \tag{15*}$$

(vi) *$\mathbf{P}$ is an OPR with*

$$x(1 + y) = x + xy \tag{16}$$

if and only if $\mathbf{Q}(\mathbf{P})$ is a BQR with (16).

(vii) *$\mathbf{P}$ is a Boolean ring if and only if $\mathbf{Q}(\mathbf{P})$ is a Boolean ring.*
In particular, the operator $\mathbf{Q}$ defines a one-to-one correspondence between

- *GBQR's with (9) and GBQR's with (9*),*
- *OPR's and BQR's,*
- *OPR's with (15) and BQR's with (15*).*

Proof:

(i) First of all, we notice that by (13) we have

$$1 \oplus y = 1 + y \quad \text{for all} \quad y \in R. \tag{13a}$$

From this follows immediately that $\mathbf{L}(\mathbf{Q}(\mathbf{P})) = \mathbf{L}(\mathbf{P})$. If we put $\mathbf{Q}(\mathbf{Q}(\mathbf{P})) = (P; \hat{+}, \cdot, 0, 1)$, then we have (by using (13a) and (13)): $x \hat{+} y = 1 \oplus (x \oplus (1 \oplus y)) = 1 + (x \oplus (1 + y)) = 1 + (1 + (x + (1 + (1 + y)))) = 1 + (1 + (x + y)) = x + y$, which shows $\mathbf{Q}(\mathbf{Q}(\mathbf{P})) = \mathbf{P}$.

(ii) Let **P** be a GBQR with (11), then we have $x \oplus y = 1 + (x + (1 + y)) = 1 + ((x + 1) + y) = 1 + (y + (1 + x)) = y \oplus x$, $x \oplus 0 = 1 + (x + (1 + 0)) = 1 + (x + 1) = 1 + (1 + x) = x$, hence **Q**(**P**) satisfies (1) and (2).
(3)–(7) for **Q**(**P**) are trivial, since $\cdot$, 0 and 1 are the same as in **P**. From (13a) we immediately get that **Q**(**P**) satisfies (8), thus **Q**(**P**) is a GBQR. Furthermore, we have by (13a) and (13): $x \oplus (1 \oplus y) = x \oplus (1 + y) = 1 + (x + (1 + (1 + y))) = 1 + (x + y)$, and by (1) we get: $(x \oplus 1) \oplus y = y \oplus (1 \oplus x) = 1 + (y + x) = 1 + (x + y)$, which proves (11), thus **Q**(**P**) is a GBQR with (11). Conversely, if **Q**(**P**) is a GBQR with (11), **Q**(**Q**(**P**)) = **P** is a GBQR with (11).

(iii) Let **P** be a GBQR with (9), then by (ii) and Lemma 2.2 **Q**(**P**) is a GBQR with (11). If we put $1 + y$ instead of y in (9), we obtain by using (13) and (4):

$$(1 + (1 + x)(1 + y))(1 + xy) = 1 + (x + (1 + y)). \qquad (9a)$$

By (13a) this implies: $(1 \oplus (1 \oplus x)(1 \oplus y))(1 \oplus xy) = (1 + (1 + x)(1 + y))(1 + xy) = x \oplus y$, i.e. (9^*). Let **Q**(**P**) be a GBQR with (9^*), then by (13a) $(1 + (1 + x)(1 + y))(1 + xy) = (1 \oplus (1 \oplus x)(1 \oplus y))(1 \oplus xy) = x \oplus y = 1 + (x + (1 + y))$, i.e. (9a) holds. If we put $1 + y$ instead of y in (9) and use (13) and (4), we obtain (9).

(iv) This follows already from Prop. 1.1 and 1.2, but in this context we give a direct proof, since by (iii) it remains only to show that **P** satisfies (10) if and only if **Q**(**P**) does. First we observe that by (13a) **P** satisfies (14) if and only if **Q**(**P**) does. Now suppose that **P** is an OPR, put $x = y$ in the equation (9^*) and use (5), (13a), (13) and (14), then $x \oplus x = (1 \oplus (1 \oplus x)(1 \oplus x))(1 \oplus xx) = (1 + (1 + x))(1 + x) = x(1 + x) = 0$. Similarly, if **Q**(**P**) is a BQR and we put $x = y$ in (9), then by using (5) and (14) $1 + (x + x) = (1 + x(1 + x))(1 + x(1 + x)) = 1 + x(1 + x) = 1 + 0$, hence by (13) $x + x = 0$.

(v) This also follows already from results in [2] and [6]: In [2] it was shown that **P** satisfies (15) if and only if **L**(**P**) is orthomodular, and in [6] it was shown that **Q**(**P**) satisfies (15^*) if and only if **L**(**Q**(**P**)) is orthomodular. Nevertheless, in this context a direct proof is of some interest. First we observe that by using (12) and (13a)

$$x \oplus xy = x(1 \oplus xy) = x(1 + xy) = x + xy. \qquad (12a)$$

Now let us start with an OPR with (15), then by (12a), (13a), (1) and (13) $(1 \oplus xy) \oplus (x \oplus xy) = (1 + xy) \oplus (x + xy) = (x + xy) \oplus (1 + xy) = 1 + ((x + xy) + (1 + (1 + xy))) = 1 + ((x + xy)$

$+xy) = 1 + x$. Conversely, let $\mathbf{Q}(\mathbf{P})$ be a BQR with (15*), then by (12a), (1) and (13) $(x + xy) + xy = (x \oplus xy) + xy = 1 \oplus ((x \oplus xy) \oplus (1 \oplus xy)) = 1 \oplus (1 \oplus x) = x$.

(vi) This follows immediately from (13a) and (12a).

(vii) If $\mathbf{R}$ is a Boolean ring, then $\mathbf{Q}(\mathbf{R}) = \mathbf{R}$, and the same holds if $\mathbf{Q}(\mathbf{R})$ is a Boolean ring. □

Corollary 2.1. The operators $\mathbf{L}$ and $\mathbf{P}$ define a one-to-one correspondence between GBQR's with (9) and BLIA's.

Proof: By Prop. 1.2 and Theorem 2.2, all we have to show is that

$$\mathbf{Q}(\mathbf{R}(\mathbf{L})) = \mathbf{P}(\mathbf{L}) \qquad (*)$$

for every BLIA $\mathbf{L}$. Let $\mathbf{P}(\mathbf{L}) = (L; +, \cdot, 0, 1)$, $\mathbf{R}(\mathbf{L}) = (L; \oplus, \cdot, 0, 1)$ and $\mathbf{Q}(\mathbf{R}(\mathbf{L})) = (L; \hat{+}, \cdot, 0, 1)$, then by (a)–(c) $x \hat{+} y = 1 \oplus (x \oplus (1 \oplus y)) = (x \oplus y^{\perp})^{\perp} = (x \vee y^{\perp}) \wedge (x \wedge y^{\perp})^{\perp} = (x \vee y^{\perp}) \wedge (x^{\perp} \vee y) = x + y$, which proves our claim. □

Remark 2.1. By (i) of Theorem 2.2 we have also:

$$\mathbf{R}(\mathbf{L}) = \mathbf{Q}(\mathbf{P}(\mathbf{L})) \qquad (**)$$

for any BLIA.

Remark 2.2. As was pointed out by H. Länger (see also [10]), one can show that in any OPR (15) is equivalent to (15*), and the same holds for any BQR.

In the following, we want to give conditions for an OPR (resp. BQR) to be a Boolean ring.

Lemma 2.4. *Let* $\mathbf{P}$ *be a GBQR, then* $\mathbf{Q}(\mathbf{P}) = \mathbf{P}$ *if and only if*

$$1 + (x + y) = (1 + x) + y \qquad (17)$$

holds in $\mathbf{P}$.

Proof: $x \oplus y = x + y$ if and only if $1 + (x + (1 + y)) = x + y$. By (13), this is equivalent to $x + (1 + y) = 1 + (x + y)$ which by (1) is equivalent to $1 + (y + x) = (1 + y) + x$. □

Lemma 2.5. *Let* $\mathbf{P}$ *be a GBQR with (9), then* $\mathbf{Q}(\mathbf{P}) = \mathbf{P}$ *if and only if* $\mathbf{R}(\mathbf{L}(\mathbf{P})) = \mathbf{P}$.

Proof: First we observe that $\mathbf{P} = \mathbf{P}(\mathbf{L}(\mathbf{P}))$, by Corollary 2.1. Now suppose that $\mathbf{Q}(\mathbf{P}) = \mathbf{P}$, then by $(**)$ we obtain $\mathbf{P} = \mathbf{Q}(\mathbf{P}(\mathbf{L}(\mathbf{P}))) = \mathbf{R}(\mathbf{L}(\mathbf{P}))$. If conversely $\mathbf{R}(\mathbf{L}(\mathbf{P})) = \mathbf{P}$, then by $(*)$ $\mathbf{Q}(\mathbf{P}) = \mathbf{Q}(\mathbf{R}(\mathbf{L}(\mathbf{P}))) = \mathbf{P}(\mathbf{L}(\mathbf{P})) = \mathbf{P}$. □

Lemma 2.6. *Let* **P** *be a GBQR with (9) and* (9^*), *then (17) holds in* **P**.

Proof: If we put $1+x$ instead of x in (9^*) and use (13), we get $(1+x(1+y))(1+(1+x)y) = (1+x)+y$, hence by (4) and (9) we get (17). □

Lemma 2.7. *Let* **P** *be a GBQR with (17), then we have:*

(i) **P** *satisfies (9) if and only if* **P** *satisfies* (9^*).
(ii) **P** *is an OPR if and only if* **P** *is a BQR.*
(iii) **P** *is an OPR with (15) if and only if* **P** *is a BQR with* (15^*).

Proof: Follows from Lemma 2.4 and Theorem 2.2. □

Theorem 2.3. *The following are equivalent for* $\mathbf{P} = (P; +, \cdot, 0, 1)$:

(i) **P** *is an OPR with (15) and (17).*
(ii) **P** *is a BQR with* (15^*) *and (17).*
(iii) **P** *is a Boolean ring.*
(iv) **P** *is a BQR with (16).*
(v) **P** *is an OPR with (16).*

Proof: By Lemma 2.7, (i) is equivalent to (ii). Next we show that (i) is equivalent to (iii). We consider $\mathbf{L}(\mathbf{P}) = (P; \vee, \wedge, 0, 1, ^{\perp})$, then (iii) holds if and only if **L**(**P**) is a Boolean algebra. By [2] and Lemma 2.4, (i) holds if and only if **L**(**P**) is an orthomodular lattice and **Q**(**P**) = **P**, and by Lemma 5 this holds if and only if **L**(**P**) is an orthomodular lattice and **P** = **P**(**L**(**P**)) = **R**(**L**(**P**)). As mentioned in Remark 1.2, this is equivalent to the property that **L**(**P**) is a Boolean algebra. (iii) is equivalent to (iv) by a result of [6]. By Theorem 2.2, (vi) and the fact that **Q**(**P**) = **P** for a Boolean ring **P**, we obtain that (iv) is equivalent to (v). □

Remark 2.3. According to Remark 2.2, one can replace (15) by (15^*) and (15^*) by (15) in (i) and (ii) of Theorem 2.3.

3. Compatible Ideals in OPR's and BQR's

Definition 3.1. Let $\mathbf{P} = (P; +, \cdot, 0, 1)$ be a GBQR and $\phi \neq I \subseteq P$.

(a) I is called an *ideal* of **P** if, for any $a, b \in I$ and $x \in P$, we have
  (i) $a + b \in I$, and
  (ii) $ax \in I$.
(b) The subset I is called *compatible* if for every unary polynomial function p over **P** and $a_1, a_2, a_3 \in I$, we have

(iii) $(1+(1+a_1)\cdot p(a_2))\cdot p(a_3)\in I$.
Let us recall that a *unary polynomial function* p on $\mathbf{P}$ is a function $p: P\to P$ such that there exist an $(n+1)$-ary term t over $\mathbf{P}$ (where n is a non-negative integer) and elements $x_1,\ldots,x_n\in P$ with $p(x)=t(x,x_1,\ldots,x_n)$ for all $x\in P$.

(c) I is called a *congruence kernel* of $\mathbf{P}$ if there exists a congruence Θ on $\mathbf{P}$ with $I=[0]\Theta$, the class of 0 with respect to Θ.

Remark 3.1. The element 0 is contained in every ideal (put $x=0$ in (ii) and use (6)) and in every compatible subset (put $p(x)=0$ in (iii) and use (6)).

Lemma 3.1. *Let* $\mathbf{P}$ *be a GBQR with (14), then every compatible subset of* $\mathbf{P}$ *is an ideal of* $\mathbf{P}$.

Proof: In (b), take $n=1, t(x,y)=x+y$, then for all $a_1,a_2,a_3\in I$ and $x_1\in P$ we have

(iv) $(1+(1+a_1)(a_2+x_1))(a_3+x_1)\in I$.

By taking $x_1=a_1$ and $a_2=0$ we get $(1+(1+a_1)a_1)(a_3+a_1)\in I$. By (14) we have $(1+a_1)a_1=0$, which yields $a_3+a_1\in I$. If we take $n=1$ and $t(x,y)=x\cdot y$ in (b), then for all $a_1,a_2,a_3\in I$ and $x_1\in P$ we have

(v) $(1+(1+a_1)(a_2x_1))(a_3x_1)\in I$,

and taking $a_2=0$ we get $a_3x_1\in I$. □

Lemma 3.2. *Let* $\mathbf{P}$ *be GBQR and I a congruence kernel of* $\mathbf{P}$*, then the following holds:*

(i) I is an ideal of $\mathbf{P}$.
(ii) If $\mathbf{P}$ *satisfies (14), then I is compatible.*

Proof:

(i) Let $I=[0]\Theta, a,b\in I$ and $x\in P$, then $a\Theta 0, b\Theta 0$, and $x\Theta x$, hence $a+b\Theta 0+0=0$ and $a\cdot x\Theta 0\cdot x=0$, thus $a+b, ax\in I$.

(ii) Let n be a non-negative integer, t an $(n+1)$-ary term of $\mathbf{P}$, $x_1,\ldots,x_n\in P$ and $a_1,a_2,a_3\in I$, then $a_i\Theta 0$ implies $(1+(1+a_1)$ $t(a_2,\ x_1,\ldots,x_n))t(a_3,\ x_1,\ldots,x_n)\Theta(1+(1+0)t(0,x_1,\ldots,x_n))$ $t(0,x_1,\ldots,x_n)=(1+t(0,x_1,\ldots,x_n))t(0,x_1,\ldots,x_n)=0$, by (14). Thus (iii) holds. □

For the proof of our next result, we need the following well known theorem of Mal'cev (sec [11]):

Proposition 3.1. Let $\mathbf{A} = (A; F)$ be an algebra and $\phi \neq I \subseteq A$. Then I is a class of some congruence on $\mathbf{A}$ if and only if for every unary polynomial function p on $\mathbf{A}$ we have: $a, b, p(a) \in I$ implies $p(b) \in I$.

Lemma 3.3. *If* $\mathbf{P}$ *is a GBQR with (14) and I a compatible ideal of* $\mathbf{P}$*, then I is a congruence kernel on* $\mathbf{P}$.

Proof: By Remark 3.1, $0 \in I$. So let p be a unary polynomial function on $\mathbf{P}$. By Prop. 3.1, we have to show: $a, b, p(a) \in I$ implies $p(b) \in I$.

We apply (iii) of Def. 3.1 for $a_1 = p(a), a_2 = a$ and $a_3 = b$, then $(1 + (1 + p(a)) \cdot p(a)) \cdot p(b) \in I$. By (14), we have $(1 + p(a)) \cdot p(a) = 0$, which yields $p(b) \in I$. □

Theorem 3.1. *Let* $\mathbf{P}$ *be a GBQR with (14) and* $\phi \neq I \subseteq P$*, then I is a congruence kernel of* $\mathbf{P}$ *if and only if I is a compatible ideal of* $\mathbf{P}$.

By Lemma 2.3 and Theorem 2.1 we get

Corollary 3.1. Let $\mathbf{P}$ be an OPR or a BQR and $\phi \neq I \subseteq P$, then I is a congruence kernel of $\mathbf{P}$ if and only if I is a compatible ideal of $\mathbf{P}$.

Theorem 3.2. *Let* $\mathbf{P}$ *be a GBQR with (14),* Θ *a congruence on* $\mathbf{P}$*, I the class of 0 and C any class with respect to* Θ*. Then* $a \in I$ *if and only if there exists an element* $c \in C$ *with* $ac = 0$ *and* $1 + (1 + a)(1 + c) \in C$*. In particular, I is uniquely determined by C.*

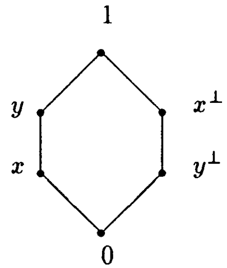
Proof:
Let $a \in P$ such that there exists an element $c \in C$ with $ac = 0$ and $d := 1 + (1 + a)(1 + c) \in C$. From $c\Theta d$ we obtain $0 = ac\,\Theta\, ad$, i.e. $ad \in I$. We have $ad = a(1 + (1 + a)(1 + c))$. From (8) with $1 + y$ instead of y and (13) we get $1 + (1 + x(1 + y))y = 1 + y$. Using (13) again, we get $(1 + x(1 + y))y = y$. Putting $x = 1 + c$ and $y = a$ and using (4), we have $ad = a \in I$.

Conversely, let $a \in I$, i.e. $a\,\Theta\, 0$. Then we obtain $(1 + a)b\Theta(1 + 0)b = b$ and $1 + (1 + (1 + a)b)(1 + a)\,\Theta\, 1 + (1 + (1 + 0)b)(1 + 0) = 1 + (1 + b) = b$ for all $b \in P$. So if $b \in C$, then $c := (1 + a)b \in C$ and $1 + (1 + a)(1 + c) = 1 + (1 + (1 + a)b)(1 + a) \in C$, too. Moreover $ac = a((1 + a)b) = (a(1 + a))b = 0 \cdot b = 0$. □

Corollary 3.2. Let $\mathbf{P}$ be a GBQR with (14), Θ a congruence on $\mathbf{P}$ and suppose that Θ has a one-element class, then $[0]\Theta = \{0\}$.

Proof: Suppose that $C = \{c\}$ is a class of Θ, then by Theorem 3.3 $a \in [0]\Theta$ if and only if $ac = 0$ and $1 + (1 + a)(1 + c) = c$. As in the first part of the proof of Theorem 3.3, we obtain $a(1 + (1 + a)(1 + c)) = a$, thus we have $a = ac = 0$. □

The following example shows that a compatible ideal can be the congruence kernel of more than one congruence:



Example. Let **L** be the ortholattice defined by the diagram above and **P** = **P**(**L**) the corresponding OPR. The partition $\{0\}$, $\{x, y\}$, $\{x^\perp, y^\perp\}$, $\{1\}$ defines a congruence Θ_1 on **L** which of course is a congruence on **P**, too. Let Θ_2 be the identity congruence on **P**, then $\Theta_1 \neq \Theta_2$ and $[0]\Theta_1 = \{0\} = [0]\Theta_2$. Furthermore, this example shows that some congruence classes can be singletons, but others not.

Remark 3.2. If **L** is an orthomodular lattice, such situations as in the preceding example cannot happen since every orthomodular lattice is congruence regular (i.e., every congruence class uniquely determines the whole congruence) and congruence uniform (i.e., all classes of a congruence have the same cardinality) (see [3] and [8]).

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