On Additive Functions Fulfilling some Additional Condition*

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Abstract

Let $D \subset \mathbb{R}^2$ be an arbitrary set. We consider the following question: What kind of assumptions on $D$ imply that every additive function $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$(x, y) \in D \Rightarrow f(x) + f(y) = 0$$

is identically equal to zero? It is true if $D$ is a non-empty open subset of $\mathbb{R}^2$. G. Szabó posed this problem for $D = \{(x, y); x^2 + y^2 = 1\}$ ([7]). We give an affirmative answer to Szabó’s question and, moreover, we give some sufficient conditions to obtain the above assertion in much more general spaces.

1. Let $X$ and $Y$ be linear spaces over the rationals $\mathbb{Q}$. A function $f : X \to Y$ is called additive if it satisfies the (Cauchy’s) equation

$$f(x + y) = f(x) + f(y), \quad x, y \in X.$$  

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Every additive function is uniquely determined by its values on a so-called Hamel base (i.e. a base of \( X \) over the rationals) and it fulfills the condition

\[
f(rx) = rf(x), \quad x \in X, \quad r \in \mathbb{Q},
\]

(cf. [4], for example). We start our considerations with two examples.

**Example 1.** Let \( f : \mathbb{R} \to \mathbb{R} \) (\( \mathbb{R} \) is here and in the sequel the set off all reals) be a discontinuous additive function vanishing on a saturated non-measurable in the Lebesgue sense subset \( S \) ([4] p. 58 and 297-Th. 7) and put \( D = (S \times \mathbb{R}) \cup (\mathbb{R} \times S) \). Evidently \( f \) is not identically equal to zero and condition (1) is fulfilled.

The set \( D \), though large in a sense, does not contain any segment. The following second example shows that even when \( D \) contains a segment it is possible to find a non-zero additive function fulfilling condition (1).

**Example 2.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a discontinuous additive function such that the restriction of \( f \) to the set of all rationals is equal to zero. If \( D = \{(x, y); \max\{|x|, |y|\} = 1\} \), then \( f \) satisfies condition (1) and is not identically equal to zero.

The set from Example 2 is a unit circle if we treat \( \mathbb{R}^2 \) as a linear space endowed with the norm \( \|(x, y)\| = \max\{|x|, |y|\} \). We shall show that the answer to our (Szabó’s) question is positive if we take a different norm in \( \mathbb{R}^2 \). We have the following

**Remark 1.** Let \( D = \{(x, y) \in \mathbb{R}^2; |x| + |y| = 1\} \) and let \( f : \mathbb{R} \to \mathbb{R} \) be an additive function fulfilling condition (1). Then \( f \) is identically equal to zero.

**Proof:** According to our assumptions we obtain

\[
f(x)f(1 - x) = 0, \quad x \in (0, 1).
\]

If \( f(1) = 0 \) it follows from (3) that \( f(x) = 0, x \in (0, 1) \), and hence \( f \equiv 0 \). Assume

\[
f(1) \neq 0
\]

and take an \( x_0 \in (0, 1) \) such that \( f(x_0) = 0 \). For arbitrary \( \varepsilon \in \mathbb{R} \) there exists an integer \( n \) such that \( \varepsilon + nx_0 \in (0, 1) \). Thus

\[
f(\varepsilon + nx_0)f(1 - \varepsilon - nx_0) = 0,
\]

which, because of the additivity of \( f \), yields the condition

\[
f(\varepsilon)f(1 - \varepsilon) = 0, \quad \varepsilon \in \mathbb{R}.
\]
Putting $-z$ instead of $z$ in this equality and adding both we get
\[ f(z)f(1) = 0, \]
which contradicts (4) and proves Remark 1.

A positive answer to Szabó’s question is contained in

**Theorem 1.** Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function fulfilling condition (1) where $D = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$. Then $f$ is identically equal to zero.

**Proof:** Take an arbitrary $x \in (0, 1)$ and choose a $y$ such that $x^2 + y^2 = 1$. Setting
\[ u = \frac{3x + 4y}{5}, \quad v = \frac{4x - 3y}{5} \]
we observe that
\[ u^2 + v^2 = x^2 + y^2 = 1. \]
By virtue of (1)
\[ f(u)f(v) = f(x)f(y) = 0. \] (5)
Moreover, by (5)
\[ 0 = f(u)f(v) = \frac{1}{25}[3f(x) + 4f(y)][4f(x) - 3f(y)] = \frac{12}{25}(f(x)^2 - f(y)^2) \]
and hence $f(x)^2 = f(y)^2$. On account of (5) $f(x) = 0$. Due to the arbitrariness of $x \in (0, 1)$, $f$ is identically equal to zero because it is additive.

**Corollary 1.** A similar result holds true if $D = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = r^2\}$, where $r > 0$ is an arbitrary constant.

**Proof:** The function $F(x) = f(rx), x \in \mathbb{R}$ fulfills all assumptions of Theorem 1. We have also

**Theorem 2.** Let $X$ be a real normed space and let $Y$ be an arbitrary linear space. If $f : X \to Y$ is an arbitrary additive function fulfilling the condition
\[ ||x||^2 + ||y||^2 = 1 \Rightarrow f(x) = 0 \text{ or } f(y) = 0, \]
then $f$ is identically equal to zero.
Proof: First, let us assume that \( \dim X > 1 \). Take an arbitrary \( x \in X \) such that \( \|x\| = \frac{\sqrt{2}}{2} \) and put \( y = x \). Then

\[
\|x\|^2 + \|y\|^2 = 1
\]

and by our assumption we get

\[
f(x) = 0,
\]

which means that \( f \) vanishes on a circle \( C = \{x \in X : \|x\| = \frac{\sqrt{2}}{2}\} \). Since \( \dim X \geq 2 \) for every \( u \in X, \|u\| < \frac{\sqrt{2}}{2} \), there exist \( v_1, v_2 \in C \) such that \( v_1 + v_2 = u \) ([1], see the proof of Lemma 1). Consequently

\[
f(u) = f(v_1 + v_2) = f(v_1) + f(v_2) = 0.
\]

Thus \( f \), being an additive function vanishing on a ball, has to be identically equal to zero.

If \( \dim X = 0 \), the assertion is trivial. If, finally, \( \dim X = 1 \) we may assume that \( X = \mathbb{R} \) and that \( \|x\| = r^{-1}|x| \) for some \( r > 0 \). Thus for every linear functional \( \varphi : Y \to \mathbb{R} \) the function \( \varphi \circ f : X \to \mathbb{R} \) satisfies the assumptions of Corollary 1, implying that \( \varphi \circ f = 0 \). But the linear functionals on \( Y \) separate the points of \( Y \). Thus \( f = 0 \).

Let \( G \) be an abelian group and let \( K \) be a field of characteristic zero. For mappings \( w : G \to K \) and an element \( h \in G \) the difference operator \( \Delta_h \) is defined by

\[
\Delta_h w(x) := w(x + h) - w(x).
\]

A mapping \( w : G \to K \) is called a generalized polynomial of degree less than \( n + 1 \) iff

\[
\Delta_h^{n+1} w(x) = 0, \quad x, h \in G,
\]

where \( \Delta^k \) denotes the \( k \)-th iterate of \( \Delta \).

**Theorem 3.** Let \( f : G \to K \) be an additive function and let

\[
D = \{(v(x), w(x)) \in K \times K; x \in G\},
\]

where \( v, w : G \to K \) are generalized polynomials such that \( \text{lin}_Q v(G) = \text{lin}_Q w(G) = K \). If \( f \) fulfills condition (1), then it is identically equal to zero.

Proof: By our assumptions

\[
f(v(x)) f(w(x)) = 0, \quad x \in G.
\]

Since \( f \circ v \) and \( f \circ w \) are generalized polynomials we can apply a result of F. Halter-Koch, L. Reich and J. Schwaiger ([3], Th. 2). Therefore
$f \circ v \equiv 0$ or $f \circ w \equiv 0$. It follows from the equality $\text{lin}_Q v(G) = \text{lin}_Q w(G) = K$ that $f$ is identically equal to zero.

**Remark 2.** The assumption $\text{lin}_Q v(G) = \text{lin}_Q w(G) = K$ is essential in Theorem 3.

This can be seen by taking $v = \text{id}$ and $w = f$, where $f$ is a function as defined in Example 2.

**Corollary 2.** Let $v, w : \mathbb{R} \to \mathbb{R}$ be arbitrary (ordinary) polynomials of degree at least one. If $f : \mathbb{R} \to \mathbb{R}$ is an additive function fulfilling condition (6) then it is identically equal to zero.

This is so, since $v(R)$ and $w(R)$ are non-trivial intervals.

Condition (1) may be generalized by replacing the righthand side of the implication (i.e. $f(x)f(y) = 0$ for $(x, y) \in D$) by $Q(f(x), f(y)) = 0$ for all $(x, y) \in D$, where $Q$ is a polynomial in indeterminates $X$ and $Y$ over $\mathbb{R}$ ($Q \in \mathbb{R}[X, Y]$). This means that we now are interested in conditions on $D \subseteq \mathbb{R}^2$ such that

$$(x, y) \in D \Rightarrow Q(f(x), f(y)) = 0 \quad (1')$$

for an additive function $f : \mathbb{R} \to \mathbb{R}$ implies $f = 0$.

In this situation we will show

**Theorem 3'.** Let $f : \mathbb{R} \to \mathbb{R}$ be additive and let $p$ and $q$ be generalized polynomials of degree 1, i.e. $p = g + a, q = b + b$, where $g, b : \mathbb{R} \to \mathbb{R}$ are additive and $a, b$ real constants. Assume that $p(\mathbb{R})$ and $q(\mathbb{R})$ contain Hamel bases. Furthermore, let $Q \in \mathbb{R}[X, Y]$ such that no polynomial $AX + BY + C$ with $AB \neq 0$ divides $Q(X, Y)$, and let

$$D : = \{(p(u), q(u)) | u \in \mathbb{R}\} \subseteq \mathbb{R}^2.$$ 

Then, if

$$(x, y) \in D \Rightarrow Q(f(x), f(y)) = 0, \quad (1')$$

we have $f = 0$.

**Proof:** We have $f(p(u)) = (f \circ g)(u) + \epsilon, f(q(u)) = (f \circ b)(u) + d$, where $\epsilon = f(a), d = f(b), f \circ g$ and $f \circ b$ are additive, and since $p(\mathbb{R}), q(\mathbb{R})$ contain Hamel bases, the same holds for $g(\mathbb{R}), b(\mathbb{R})$, and so $f \circ g \neq 0, f \circ b \neq 0$. By (1') we have

$$Q((f \circ g)(u) + \epsilon, (f \circ b)(u) + d) = 0, \quad u \in \mathbb{R}.$$ 

We denote by $Q_1(X, Y)$ the polynomial $Q_1(X, Y) := Q(X + \epsilon, Y + d)$, where $Q_1 \neq 0, Q_1((f \circ g)(u), (f \circ b)(u)) = 0, u \in \mathbb{R}$.
By [6, theorem 1] we get that $f \circ g, f \circ h$ are linearly dependent over $\mathbb{R}$, i.e. there exists $(\lambda, \mu) \in \mathbb{R}^2, (\lambda, \mu) \neq (0, 0)$ such that

$$\lambda(f \circ g) + \mu(f \circ h) = 0. \quad (7)$$

Since $f \circ g \neq 0, f \circ h \neq 0$ we deduce that $\lambda \neq 0, \mu \neq 0$. But then by [6, theorem 2] we see that

$$\lambda X + \mu Y|_{Q_1(X, Y)},$$

and therefore

$$\lambda X + \mu Y - (\lambda c + \mu d)|_{Q(X, Y)},$$

where $\lambda \mu \neq 0$, which contradicts the assumption of the theorem. So we have necessarily $f = 0$, which concludes the proof.

The set $D$ from Example 1 is large in a certain sense; it is saturated non-measurable in the Lebesgue sense as well as it is a second category set without Baire property. However, we prove the following

**Theorem 4.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be an additive function fulfilling condition (1) and assume that $D \subset \mathbb{R}^{2n}$ is a Lebesgue measurable subset with positive measure. Then $f$ is identically equal to zero.

**Proof:** The set

$$H := \{x \in \mathbb{R}^n; f(x) = 0\}$$

is a subgroup of $\mathbb{R}^n$ and since $D \subset (H \times \mathbb{R}^n) \cup (\mathbb{R}^n \times H)$ the outer Lebesgue measure of $H$ is positive. It is not hard to check that $H$ is dense in $\mathbb{R}^n$. By Smítal’s lemma ([4], [5]) the set $G := (H \times H) + D$ is of full Lebesgue measure in $\mathbb{R}^{2n}$ (in fact; since $\mathbb{R}^n$ is separable there exists a countable subset $H_0$ of $H$ which is dense in $\mathbb{R}^n$, and by Smítal’s lemma the set $(H_0 \times H_0) + D$ has full Lebesgue measure in $\mathbb{R}^{2n}$ and, of course, $(H_0 \times H_0) + D \subset G$). Moreover, for every $(x, y) \in G$ we have $x = b_1 + d_1, y = b_2 + d_2, b_1, b_2 \in H, (d_1, d_2) \in D$, and hence $f(x)f(y) = f(d_1)f(d_2) = 0$. Therefore

$$G \subset (H \times \mathbb{R}^n) \cup (\mathbb{R}^n \times H) =: S.$$  

We will show that $H$ is measurable in the Lebesgue sense and of the full measure in $\mathbb{R}^{2n}$. By Fubini’s theorem the set

$$B := \{x \in \mathbb{R}^n; S_x = \{y \in \mathbb{R}^n; (x, y) \in S\} \text{ is measurable}\}$$

is measurable in the Lebesgue sense and of full measure in $\mathbb{R}^n$. If $B \subset H$, then $H$ is measurable and of the full measure in $\mathbb{R}^n$. If $B \setminus H \neq \emptyset$, take an $x \in B \setminus H$. Then $S_x = H$ and $x \in B$. So, $H$ is measurable, too. Thus $H$,
being a dense subgroup of full measure in \( \mathbb{R}^n \), is equal to \( \mathbb{R}^n \). (In fact, any subgroup of \( \mathbb{R}^n \) of positive measure equals \( \mathbb{R}^n \): Assume that \( H \) is a full measure group in \( \mathbb{R}^n \). Take an arbitrary \( x \) from \( \mathbb{R}^n \). Then the set \( x - H \) is also full measure in \( \mathbb{R}^n \) and therefore by the Steinhaus theorem the intersection \( H \cap (x - H) \) is a nonempty set. Choosing a \( z \) from this intersection we get that \( x = z + (x - z) \) belongs to \( H + H = H \). Thus \( H = \mathbb{R}^n \).)

The proof of Theorem 4 is finished.

A topological analogue of Theorem 4 is also true. One can prove the following

**Theorem 5.** Let \( D \) be a second category subset of \( \mathbb{R}^{2n} \) with the Baire property and let \( f : \mathbb{R}^n \to \mathbb{R} \) be an additive function fulfilling condition (1). Then \( f \) is identically equal to zero.

**Proof:** The proof is quite similar to the proof of Theorem 4 because Fubini’s theorem and Smital’s lemma have topological analogues ([2], [4]).

The results of Remark 1 and Theorem 1 can be viewed as special cases of the following.

**Theorem 6.** Let \( u, v : T \to \mathbb{R} \) be such that for all \( t \in T \) there is some \( t_1 \in T \) and some \( 2 \times 2 \)-matrix \( Q \) with rational and nonvanishing entries \( a, b, c, d \) such that \( (u(t_1), v(t_1))^T = Q(u(t), v(t))^T \). Moreover let \( u(T) \) or \( v(T) \) generate \( \mathbb{R} \) as a \( \mathbb{Q} \)-vector space. Then we have that the condition

\[
(f \circ u) \cdot (f \circ v) = 0
\]

implies \( f = 0 \).

**Proof:** Fix \( t \in T \). Without loss of generality we may suppose that \( f(u(t)) = 0 \). Choosing \( t_1 \) and \( Q \) as above and using the fact that \( f(u(t_1)) \cdot f(v(t_1)) = 0 \) we get

\[
0 = f(au(t) + bv(t))f(cu(t) + dv(t))
\]

\[
= acf(u(t))^2 + adf(u(t))f(v(t)) + bcf(v(t))f(u(t)) + bd f(v(t))^2
\]

\[
= bd f(v(t))^2,
\]

implying that \( f(v(t)) = 0 \). Thus \( f \circ u = f \circ v = 0 \) which gives us the desired result.

**Remark 3.** Using \( u = \cos \) and \( v = \sin \) we get Theorem 1 with \( t_1 = t + t_0 \) where \( t_0 \) is such that \( \cos(t_0) = 3/5 \) and \( \sin(t_0) = 4/5 \), for example. Remark 1 may be considered as the case \( T = ]0, 1[, u(t) = t, v(t) = 1 - t, a = b = c = d = 1/2 \).
A different example (hyperbola) is given by \( u = \cosh, \ v = \sinh, \ \tau_1 = \tau + \tau_0 \), where now \( \tau_0 \) is chosen in such a way that both \( \cosh(\tau_0) \) and \( \sinh(\tau_0) \) are positive rationals (which of course is possible).

References


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