Sequential Compactness in Constructive Analysis

By

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Abstract

A new constructive notion of sequential compactness is introduced, and its relation to completeness and totally boundedness is explored.

In this note we complement the work in [3] by introducing, within the framework of (Bishop’s) constructive mathematics [1], a new approach to sequential compactness. We begin with the fundamental definition on which the paper is based.

A sequence $\mathbf{x} = (x_n)$ in a metric space $(X, \rho)$ has at most one cluster point if the following condition holds:

There exists $\delta_x > 0$ such that if $0 < \delta < \delta_x$ and $\rho(a, b) > 2\delta$, then either $\rho(x_n, a) > \delta$ for all sufficiently large $n$ or else $\rho(x_n, b) > \delta$ for all sufficiently large $n$.

Note that each subsequence of $(x_n)$ then has at most one cluster point: indeed, the same $\delta_x$ works for such a subsequence as for the original sequence $\mathbf{x}$.

A Cauchy sequence $\mathbf{x}$ has at most one cluster point. To see this, let $\rho(a, b) > 2\delta > 0$. Choose $\varepsilon > 0$ such that $\rho(a, b) > 2(\delta + \varepsilon)$, and then choose $N$ such that $\rho(x_n, x_m) < \varepsilon$ for all $m, n \geq N$. Since

$$(\rho(x_N, a) - \delta - \varepsilon) + (\rho(x_N, b) - \delta - \varepsilon) \geq \rho(a, b) - 2(\delta + \varepsilon) > 0,$$

the sequence $\mathbf{x}$ is not Cauchy. Therefore, $\mathbf{x}$ cannot have a convergent subsequence, and hence it cannot have a cluster point.
either $\rho(x_N, a) > \delta + \varepsilon$ or $\rho(x_N, b) > \delta$. In the first case, $\rho(x_n, a) > \delta$ for all $n \geq N$; in the second, $\rho(x_n, b) > \delta$ for all $n \geq N$.

We call $X$ **sequentially compact** if every sequence in $X$ that has at most one cluster point converges to a limit in $X$. To see that this notion of sequential compactness is classically equivalent to the usual one, suppose that $X$ is sequentially compact in our sense, and let $(x_n)$ be any sequence in $X$; if $(x_n)$ does not have a cluster point, then it has at most one cluster point and so converges in $X$, a contradiction. On the other hand, suppose that $X$ is sequentially compact in the usual sense, and consider a sequence $(x_n)$ in $X$ that has at most one cluster point. Since $X$ is classically sequentially compact, there exists a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)$ that converges to a limit $x_\infty$ in $X$. If $(x_n)$ does not converge to $x_\infty$, then there exists a subsequence of $(x_n)$ that is bounded away from $x_\infty$; this subsequence has cluster points, but none of those can equal $x_\infty$; this contradicts our hypothesis that $(x_n)$ has at most one cluster point.

Classically, a metric space is sequentially compact if and only if it is complete and totally bounded ([4], (3.16.1)). There is a natural approximate interval-halving proof that $[0,1]$ is constructively sequentially compact in our sense. Given a sequence $(x_n)$ in $[0,1]$ that has at most one cluster point, let $I_0 = [0,1]$. Taking $a = \frac{1}{2}$ and $b = \frac{3}{4}$ in the definition of at most one cluster point, we see that as $|a-b| > \frac{2}{5}$,

- either $|x_n - \frac{1}{2}| > \frac{1}{5}$, and therefore $x_n > \frac{2}{5}$, for all sufficiently large $n$;
- or else $|x_n - \frac{3}{4}| > \frac{1}{5}$, and therefore $x_n < \frac{3}{5}$, for all sufficiently large $n$.

In the first case, take $I_1 = [\frac{3}{5}, 1]$; in the second, take $I_1 = [0, \frac{3}{5}]$. Carrying on in this way, we produce closed intervals $I_0 \supset I_1 \supset I_2 \supset \ldots$ such that for each $n$, $|I_n| = \frac{3}{5}|I_{n-1}|$ and $x_k \in I_n$ for all sufficiently large $k$. Then there exists a unique point $x_\infty \in \bigcap_{n=0}^\infty I_n$, and it is routine to show that $x_\infty = \lim_{n \to \infty} x_n$.

The following key lemma will enable us to generalise this from $[0,1]$ to any complete, totally bounded metric space.

**Lemma 1.** Let $x = (x_n)$ be a sequence with at most one cluster point in a metric space $X$, let $\delta_x$ be as in the foregoing definition, and let $0 < \varepsilon < \delta_x$. Suppose that there exists a finitely enumerable set $F$ of $X$ such that for each $n$ there exists $x \in F$ with $\rho(x, x_n) < \varepsilon$. Then $\rho(x_m, x_n) < 8\varepsilon$ for all sufficiently large $m$ and $n$.

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1 The classical property of sequential compactness does not hold constructively even for the pair set $\{0,1\}$, and so is constructively useless.

2 A set is **finitely enumerable** if it is the range of a mapping $f$ from $\{1, \ldots, n\}$, for some natural number $n$. If also $f$ is one–one, then its range is said to be **finite**.
Proof: Let $\xi_1 \in F$. Either $\rho(\xi, \xi_1) < 3\varepsilon$ for all $\xi \in F$ or else there exists $\xi' \in F$ such that $\rho(\xi', \xi_1) > 2\varepsilon$. In the first case we have $\rho(x_n, \xi_1) < 4\varepsilon$ for all $n$, and therefore $\rho(x_m, x_n) < 8\varepsilon$ for all $m$ and $n$; so we may assume that the second case obtains. Accordingly, by our hypothesis on $x$, either $\rho(x_n, \xi_1) > \varepsilon$ for all sufficiently large $n$ or else $\rho(x_n, \xi_1') > \varepsilon$ for all sufficiently large $n$. Interchanging $\xi_1$ and $\xi'$, if necessary, we may assume that $\rho(x_n, \xi_1) > \varepsilon$ for all $n \geq N_1$. If follows that for each $n \geq N_1$ there exists

$$\xi \in F \sim \{\xi_1\} = \{x \in F : x \neq \xi_1\}$$

such that $\rho(x_n, \xi) < \varepsilon$. We may therefore repeat the foregoing argument, with $x$ replaced by $(x_n)_{n \geq N_1}$ and $F$ replaced by $F \sim \{\xi_1\}$. In this way we obtain $\xi_2 \in F \sim \{\xi_1\}$ such that

- either $\rho(x_n, \xi_2) < 4\varepsilon$ for all $n \geq N_1$, and therefore $\rho(x_m, x_n) < 8\varepsilon$ for all $m, n \geq N_1$,
- or else there exists a positive integer $N_2 > N_1$ such that $\rho(x_n, \xi_2) > \varepsilon$ for all $n \geq N_2$.

Executing this procedure a total of at most $\#F$ times, we are guaranteed to produce $N$ such that $\rho(x_m, x_n) < 8\varepsilon$ for all $m, n \geq N$. Q.E.D.

Corollary 2. If $X$ is a totally bounded metric space, then any sequence in $X$ with at most one cluster point is a Cauchy sequence.

Corollary 3. The following are equivalent conditions on a sequence $(x_n)$ in any metric space $X$:

(i) $(x_n)$ is totally bounded and has at most one cluster point.

(ii) $(x_n)$ is a Cauchy sequence.

The following constructive generalisation of the Bolzano-Weierstraß Theorem is an immediate consequence of Corollary 2.

Theorem 4. A complete, totally bounded metric space is sequentially compact.

We now prove some partial converses of this theorem.

Proposition 5. If $X$ is sequentially compact, then it is complete.

Proof: Every Cauchy sequence in $X$ has at most one cluster point and so converges. Q.E.D.

Proposition 6. Let $X$ be sequentially compact, and let $a$ be a point of $X$ such that for all positives, $t$ with $s < t$, either $\rho(x, a) < t$ for all $x \in X$ or else $\rho(x, a) > s$ for some $x \in X$. Then $X$ is bounded.
Proof: Construct an increasing binary sequence \((\lambda_n)\) such that
\[ \vdash \text{if } \lambda_n = 0, \text{then there exists } x \in X \text{ such that } \rho(x, a) > n, \]
\[ \vdash \text{if } \lambda_n = 1, \text{then } \rho(x, a) < n + 1 \text{ for all } x \in X. \]

We may assume that \(\lambda_1 = 0\). If \(\lambda_n = 0\), choose \(x_n \in X\) such that
\[ \rho(x_n, a) > n; \text{ if } \lambda_n = 1, \text{ set } x_n = x_{n-1}. \]
To prove that \(x = (x_n)\) has at most one cluster point, let \(\rho(y, z) > 2\delta > 0\), and choose a positive integer
\[ N > \max \{\rho(a, y), \rho(a, z)\} + \delta. \]
If \(\lambda_N = 1\), then \(x_n = x_N\) for each \(n \geq N\), so that either \(\rho(x_n, y) > \delta\) for all \(n \geq N\) or else \(\rho(x_n, z) > \delta\) for all \(n \geq N\). Consider, on the other hand, what happens if \(\lambda_n = 0\). If \(n \geq N\) and \(\lambda_n = 0\), then \(\rho(x_n, a) > n \geq N\), so
\[ \rho(x_n, y) \geq \rho(x_n, a) - \rho(a, y) > \delta \]
and likewise \(\rho(x_n, z) > \delta\). If \(n \geq N\) and \(\lambda_n = 1\), then there exists
\[ k \in \{N + 1, \ldots, n\} \text{ such that } \lambda_k = 1 - \lambda_{k-1}; \text{ whence } x_n = x_{n-1} = \cdots = x_{k-1} \text{ where, as above, } \rho(x_{k-1}, y) > \delta \text{ and } \rho(x_{k-1}, z) > \delta. \]
Thus \(x\) has at most one cluster point in \(X\) and therefore converges to a limit \(x_\infty \in X\). Choosing a positive integer \(n > 1 + \rho(x_\infty, a)\) such that
\[ \rho(x_n, x_\infty) < 1, \]
we see that \(\lambda_n = 1\). Q.E.D.

The constructive least-upper-bound principle states that if the non-empty subset \(S\) of \(R\) is not only bounded above, but also located — in the sense that for all \(\alpha, \beta\) with \(\alpha < \beta\), either \(\beta\) is an upper bound of \(S\) or else there exists \(x \in S\) with \(x > \alpha\)—then \(\sup S\) exists. The locatedness condition cannot be dropped constructively, although it is redundant classically.

Corollary 7. Under the hypotheses of Proposition 6, \(\sup_{x \in X} \rho(x, a)\) exists.

Proof: Since \(X\) is bounded by Proposition 6, we can apply the least-upper-bound principle to the set \(\{\rho(x, a) : x \in X\}\). Q.E.D.

Proposition 8. Let \(X\) be separable and sequentially compact. Then the following conditions are equivalent.
(i) For each \(\xi \in X\), \(\sup_{x \in X} \rho(x, \xi)\) exists.
(ii) \(X\) is totally bounded.

Proof: Let \((a_n)_{n=1}^\infty\) be a dense sequence in \(X\), and let \(\varepsilon > 0\). Set \(n_0 = 1\), assume (i), and construct an increasing binary sequence \((\lambda_k)_{k=1}^\infty\), and an increasing sequence \((n_k)_{k=1}^\infty\) of positive integers, such that
\[ \vdash \text{if } \lambda_k = 0, \text{then } \rho(a_{n_k}, \{a_1, a_2, \ldots, a_{n_k-1}\}) > \varepsilon, \]
\[ \vdash \text{if } \lambda_k = 1, \text{then } \sup_{x \in X} \rho(x, \{a_1, a_2, \ldots, a_{n_k-1}\}) < 2\varepsilon. \]
If \( \lambda_k = 0 \), put \( x_k = a_{n_k} \); if \( \lambda_k = 1 \), put \( x_k = x_{k-1} \). We show that the sequence \( x = (x_k)_{k=1}^{\infty} \) has at most one cluster point in \( X \). To this end, let \( 0 < \delta < \varepsilon \) and \( \rho(y, z) > 2\delta \), and choose \( j \) such that \( \rho(y, a_j) < \varepsilon - \delta \). Either \( \lambda_k = 1 \) for some \( k \leq j \), or else \( \lambda_j = 0 \). In the first case the sequence \( x \) is eventually constant and so clearly has at most one cluster point. In the second we may assume that \( \lambda_{j+1} = 0 \); so if \( i \geq j + 1 \) and \( \lambda_i = 0 \), then

\[
\rho(y, x_i) = \rho(y, a_n) - \rho(a_j, a_n) > \varepsilon - (\varepsilon - \delta) = \delta.
\]

It follows that if \( i > j + 1 \) and \( \lambda_i = 1 \), then, as \( x_i = x_k \) for some \( k \in \{ j + 1, \ldots, i - 1 \} \) with \( \lambda_k = 0 \), we also have \( \rho(y, x_i) > \delta \). This completes the proof that \( x \) has at most one cluster point.

Since \( X \) is sequentially compact, \( x \) converges to a limit \( x_\infty \in X \). Choose \( \kappa \) such that \( \rho(x_\infty, x_k) < \varepsilon/2 \) for all \( k \geq \kappa \). Then either \( \lambda_\kappa = 1 \) or else \( \lambda_\kappa = 0 \); in the latter case, as \( \rho(x_{\kappa+1}, x_\kappa) < \varepsilon \), we must have \( \lambda_{\kappa+1} = 1 \). Hence \( \{a_1, a_2, \ldots, a_{n_{\kappa+1}}\} \) is an \( \varepsilon \)-approximation to \( X \). This completes the proof that (i) implies (ii).

If, conversely, (ii) holds, then the uniform continuity of the mapping \( x \mapsto \rho(x, \xi) \) ensures that \( \sup_{x \in X} \rho(x, \xi) \) exists ([1], page 94, (4.3)). Q.E.D.

It it tempting to try working with a simpler notion of “\( x \) has at most one cluster point”: namely, that if \( a, b \) are distinct points of \( X \), then either \( x \) is eventually bounded away from \( a \), or \( x \) is eventually bounded away from \( b \). However, Specker’s Theorem ([5]; see also [2], page 58) shows that in the recursive model of constructive mathematics there exists a sequence in \([0, 1]\) which is eventually bounded away from any given recursive real number and, a fortiori, cannot converge.

References


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