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The Experimental Significance of Quantized Phase Differences

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Abstract

S. Yu [1] has shown that the phase difference between the quantum states of two oscillators is quantized, even if it is restricted to the half open interval $[0, 2\pi)$. In this note it shall be shown that the proper states of the phase correspond to a beat of two oscillations. Such beats could be realized within photon interferometry.

1. Introduction

Two bosonic oscillators with the circular frequencies ω_1 and ω_2, the creation and the annihilation operators $a_1^\dagger$, $a_2^\dagger$, a_1 and a_2 do not interact

$$[a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0, \quad [a_j, a_k^\dagger] = \delta_{jk}. \quad (j, k = 1 \text{ or } 2) \tag{1}$$

The operators N_j for the occupation numbers n_j and the Hamiltonians H_j commute

$$N_j = a_j^\dagger a_j, \quad H_j = (N_j + 1/2)\hbar\omega_j, \quad N = N_1 + N_2, \quad H = H_1 + H_2. \tag{2}$$

Thus they have a common base of proper states

$$|n_1, n_2\rangle = (n_1!n_2!)^{-1/2}(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}|0,0\rangle \exp[i(n_1\varphi_1 + n_2\varphi_2)]$$
$$\langle n_1, n_2| = (n_1!n_2!)^{-1/2}\langle 0,0|a_1^{n_1}a_2^{n_2} \exp[-i(n_1\varphi_1 + n_2\varphi_2)] \qquad (3)$$

with the orthogonality relations

$$\langle n_1', n_2'|n_1, n_2\rangle = |n_1', n_2'\rangle^\dagger |n_1, n_2\rangle = \delta(n_1, n_1')\delta(n_2, n_2'). \qquad (4)$$

The δ-sign is the Kronecker symbol, because all occupation numbers are integers

$$n_1 = 0, 1, 2, \dots, \quad n_2 = 0, 1, 2, \dots, \quad n = n_1 + n_2 = 0, 1, 2, \dots.$$

The bare vacuum state $|0,0\rangle$ has the properties

$$a_j|0,0\rangle = 0, \quad \langle 0,0|a_j^\dagger = 0, \quad \langle 0,0|0,0\rangle = 1. \quad (j = 1 \text{ or } 2) \qquad (5)$$

The occupation numbers n_j and the energies W_j are the proper values of the operators (2)

$$(N_1 - n_1)|n_1, n_2\rangle = (H_1 - W_1)|n_1, n_2\rangle = 0, \quad W_1 = (n_1 + 1/2)\hbar\omega_1$$
$$(N_2 - n_2)|n_1, n_2\rangle = (H_2 - W_2)|n_1, n_2\rangle = 0, \quad W_2 = (n_2 + 1/2)\hbar\omega_2$$
$$(N - n)|n_1, n_2\rangle = (H - W)|n_1, n_2\rangle = 0, \qquad W = W_1 + W_2. \qquad (6)$$

The relations (1), (2), (4), (5) and (6) are invariant with respect to the transformations

$$a_j \to a_j \exp(i\varphi_j), \quad a_j^\dagger \to a_j^\dagger \exp(-i\varphi_j). \quad (j = 1 \text{ or } 2)$$

Therefore, in the general case the state vectors (3) contain a phase factor of modulus 1.

For an isolated oscillator the corresponding phase φ_j is not a measurable quantity, because the spectrum of proper values

$$a_j^\dagger a_j|n_1, n_2\rangle = n_j|n_1, n_2\rangle \qquad (n_j = 0, 1, 2, \dots)$$
$$a_j a_j^\dagger|n_1, n_2\rangle = (n_j + 1)|n_1, n_2\rangle \quad (n_j + 1 = 1, 2, 3, \dots)$$

is different for the operators $a_j^\dagger a_j$ and $a_j a_j^\dagger$. Thus the operators a_j and $a_j^\dagger$ are not unitary and the phase φ_j cannot be represented by a Hermite operator. φ_j and n_j are not canonically conjugate variables.

As the phase of an isolated oscillator is not a measurable quantity, in the state vectors (4) the phase factors can be put equal to 1 for the occupation numbers $(n_1, n_2) = (0, n)$ and $(n, 0)$, if the total occupation number $n = n_1 + n_2$ is kept constant. This means that the phase differences are reduced to a finite set of numbers

$$\begin{aligned} n\varphi_1 &= 2\pi m_1, \qquad m_1 = 0, 1, 2, \ldots, n-1. \\ \Theta_{mn} &= \varphi_2 - \varphi_1 = 2\pi m/n \\ n\varphi_2 &= 2\pi m_2, \qquad m_2 = 0, 1, 2, \ldots, n-1. \\ m &= m_2 - m_1 = 0, 1, 2, \ldots, n-1. \end{aligned} \tag{7}$$

If all possible values $n = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots, n-1$ are considered, then the quantity $(\Theta_{mn}/2\pi)$ is any rational number m/n in the interval $[0, 1)$. Thus the proper values of the phase difference lie neither discrete like the energies in bound states, nor continuously like the energies of a free particle, but they lie dense. As Yu has pointed out, this is a new type and a new set of proper values for an observable.

From Eqs. (3) and (7) one gets n sets of orthogonal and normalized state vectors

$$\begin{aligned} |n-k, k; m\rangle &= [(n-k)!k!]^{-1/2}(a_1^\dagger)^{n-k}(a_2^\dagger)^k|0, 0\rangle \exp(ik\Theta_{mn}) \\ \langle n-j, j; m|n-k, k; m\rangle &= \delta_{jk} \qquad (j, k = 0, 1, 2, \ldots, n) \\ \Theta_{mn} &= 2\pi m/n. \qquad (m = 0, 1, 2, \ldots, n-1) \end{aligned} \tag{8}$$

Each of the n sets is characterized by a certain phase Θ_{mn}. In a special set the $n+1$ quantum states are proper states of the operators N_1, N_2 and N, but they are not proper states of the phase difference φ_{12}. The phase operator φ_{12} and the operator N are canonically conjugate variables. n of the $n+1$ states in the m-th set have to be added with the same amplitude, to get a proper state of the phase operator; but then the partial occupation numbers are undefined.

2. The Operator for a Phase Difference

From the matrix element

$$\langle n-k+1, k-1; m|a_1^\dagger a_2|n-k, k; m\rangle = [(n-k+1)k]^{1/2} \exp(i\Theta_{mn})$$

one can see that the operator φ_{12} is given by the equation

$$\begin{aligned} \exp(i\varphi_{12}) = (a_1^\dagger a_1)^{-1/2} a_1^\dagger a_2 (a_2^\dagger a_2)^{-1/2}(1 - |n, 0\rangle\langle n, 0|) \\ + |1, n-1\rangle\langle n, 0|(a_1^\dagger a_1)^{-1/2} a_1^\dagger a_2 |n-1, 1\rangle\langle n, 0|, \end{aligned} \tag{9}$$

if the total occupation number n is fixed. In the more general case the projection operators $|n, 0\rangle\langle n, 0|$ and $|0, n\rangle\langle n, 0|$ have to be summed up $(n = 0, 1, 2, \ldots)$. Equation (9) takes into account that no phase factor can be determined from the relations

$$a_1^\dagger a_2|n,0\rangle = 0, \qquad \langle n,0|a_1^\dagger a_2 = 0.$$

With respect to their phase factors the quantum states $|0,n\rangle$ and $|n,0\rangle$ correspond to each other. Therefore, these two states have to be treated separately. They give rise to two proper states of the phase.

In the proper states

$$|\Theta_{mn}\rangle = n^{-1/2}\sum_{k=0}^{n-1}|n-k,k;m\rangle \quad (m=0,1,2,\ldots,n-1) \tag{10}$$

of the operator φ_{12} to the proper values Θ_{mn} the phase differences are well-defined

$$[\exp(i\varphi_{12})]|\Theta_{mn}\rangle = [\exp(i\Theta_{mn})]|\Theta_{mn}\rangle, \quad (0 \le \Theta_{mn} < 2\pi) \tag{11}$$

but the special occupation numbers $n-k$ and k are undefined. The state vectors (10) form a set of orthogonal and normalized states

$$\langle\Theta_{mn}|\Theta_{m'n}\rangle = \frac{1}{n}\sum_{k=0}^{n-1}\exp\left(2\pi i k\,\frac{m-m'}{n}\right) = \delta(m,m'). \tag{12}$$

They are proper states of the total occupation number

$$(N-n)|\Theta_{mn}\rangle = 0,$$

but they are not proper states of the operators N_1, N_2, H_1, H_2 and H. Here only the expectation values are defined; they are given by the following expressions

$$\langle N_1\rangle = \frac{1}{2}(n+1), \quad \langle N_2\rangle = \frac{1}{2}(n-1), \quad \langle H_1\rangle = \left(\frac{1}{2}n+1\right)\hbar\omega_1$$
$$\langle H_2\rangle = \frac{1}{2}n\hbar\omega_2, \quad \langle H\rangle = \frac{1}{2}n\hbar(\omega_1+\omega_2)+\hbar\omega_1 \text{ for } \Theta=\Theta_{mn}. \tag{13}$$

On the other hand, if the operator φ_{12} is represented by the expression

$$\exp(i\tilde{\varphi}_{12}) = (a_1^\dagger a_1)^{-1/2}a_1^\dagger a_2(a_2^\dagger a_2)^{-1/2}(1-|n-1,1\rangle\langle n-1,1|)$$
$$+\,|0,n\rangle\langle n-1,1|(a_1^\dagger a_1)^{-1/2}a_1^\dagger a_2(a_2^\dagger a_2)^{-1/2}|n-2,2\rangle\langle n-1,1|, \tag{14}$$

then the proper states of the phase are given by the equations

$$|\tilde{\Theta}_{mn}\rangle = \sum_{k=1}^{n}|n-k,k;m\rangle, \quad \langle\tilde{\Theta}_{mn}|\tilde{\Theta}_{m'n}\rangle = \delta_{mm'}$$
$$[\exp i\tilde{\varphi}_{12}]|\tilde{\Theta}_{mn}\rangle = [\exp(i\Theta_{mn})]|\Theta_{mn}\rangle. \tag{15}$$

In this case the expectation values

$$\langle N_1 \rangle = \frac{1}{2}(n-1), \quad \langle N_2 \rangle = \frac{1}{2}(n+1), \quad \langle H_1 \rangle = \frac{1}{2} n\hbar\omega_1$$
$$\langle H_2 \rangle = \left(\frac{1}{2}n+1\right)\hbar\omega_2, \quad \langle H \rangle = \frac{1}{2} n\hbar(\omega_1+\omega_2) + \hbar\omega_2 \text{ for } \Theta = \Theta_{mn}. \tag{16}$$

are the same as the quantities (13), if the oscillators 1 and 2 are exchanged.

The generalization to more than two oscillators is possible. For example, in the case of three oscillators the normalized quantum states

$$|n_1, n_2, n_3; m_{12}, m_{31}\rangle = (n_1!n_2!n_3!)^{-1/2}(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}(a_3^\dagger)^{n_3} \times |0,0,0\rangle \exp[(2\pi i/n)(m_{12}n_2 - n_3 m_{31})] \tag{17}$$

depend on the numbers n_1, n_2, n_3, m_{12} and m_{31}, where

$$n = n_1 + n_2 + n_3, \quad 0 \le n_j \le n, \quad 0 \le m_j < n. \quad (j = 1, 2, 3) \tag{18}$$

If three integers m_1, m_2 and m_3 are chosen in agreement with the conditions (18), then the numbers

$$m_{jk} = m_k - m_j \text{ or } m_k - m_j + n \text{ for } j = 1, 2, 3$$

can be determined in such a way that the inequalities

$$0 \le m_{jk} < n$$

are satisfied. In this way the numbers m_{jk} are found unambiguously. Only in such a case, where the numbers n and n_j are even integers, there the transition

$$m_j \to m_j \pm n/2$$

does not change the phase factor $\exp(2\pi i n_j m_j/n)$. Therefore, there are more possibilities for the numbers m_{jk}

$$m_{jk} - (m_k - m_j) = 0, \pm n/2, n \text{ for even } n \text{ and } n_j. \tag{19}$$

This ambiguity has the consequence that the product of three phase factors can become negative, too, if the corresponding operators are transformed into each other by a cyclic permutation of the oscillator numbers 1, 2 and 3. This can be shown, if in application to state vector

$|n_j, n_k\rangle$, the operator (9) becomes generalized to the expression

$$\begin{aligned}\exp(i\varphi_{jk}) = {} & (a_j^\dagger a_j)^{-1/2} a_j^\dagger a_k (a_k^\dagger a_k)^{-1/2} (1 - |n_j + n_k, 0\rangle\langle n_j + n_k, 0|) \\ & + |1, n_j + n_k - 1\rangle\langle n_j + n_k, 0|(a_j^\dagger a_j)^{-1/2} \\ & \times a_j^\dagger a_k |n_j + n_k - 1, 1\rangle\langle n_j + n_k, 0|. \end{aligned} \tag{20}$$

Then the product $I(1,2,3)$ has the proper values ± 1

$$\begin{aligned} & I(1,2,3) = [\exp(i\varphi_{12})][\exp(i\varphi_{23})][\exp(i\varphi_{31})] \\ & I(1,2,3)|n_1, n_2, n_3; m_{12}, m_{31}\rangle \\ & \quad = \{\exp[(2\pi i/n)(m_{12} + m_{23} + m_{31})]\}|n_1, n_2, n_3; m_{12}, m_{31}\rangle \\ & \quad = \pm|n_1, n_2, n_3; m_{12}, m_{31}\rangle. \end{aligned} \tag{21}$$

The negative sign implies the condition that the number $n = n_1 + n_2 + n_3$ and thus at least one of the three numbers n_1, n_2 and n_3 are even integers. S. Yu has given a general proof for the proper values ± 1 of the operator $I(1,2,3)$ [1]. This operator contains projection operators; it is idempotent ($I = I^2 = I^3 = \ldots$). Therefore, any power of I again has the proper values ± 1.

3. Photon States

Now the general relations for quantum states shall be applied to photon interferometry. For the sake of simplicity n photons with the same polarization are considered. They are moving in the positive z-direction. The scalar potential vanishes. The vector potential $\boldsymbol{A}$ and the electric field strength $\boldsymbol{E}$, both shall have only x-components A_x and E_x. Then the magnetic induction $\boldsymbol{B}$ and the Poynting vector $\boldsymbol{S}$ only have a non-trivial y-component B_y and a z-component S_z, respectively

$$E_x = -\dot{A}_x, \quad B_y = \frac{\partial}{\partial z} A_x, \quad S_x = \mu_0^{-1} E_x B_y. \tag{22}$$

μ_0 is the induction constant. The energy density w is given by the equation

$$2\mu_0 \mathrm{c}^2 w = E_x^2 + \mathrm{c}^2 B_y^2, \quad \mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{Vs/Am}. \tag{23}$$

c is the velocity of light. If the plane waves $\exp[i\omega(z/\mathrm{c} - t)]$ are normalized to the volume V_0, then the operators for the electromagnetic observables are given by the following expressions

$$
\begin{aligned}
A_x &= \mathrm{c}(\mu_0\hbar/2V_0)^{1/2}\sum_j \omega_j^{-1/2}\{a_j \exp[i\omega_j(z/\mathrm{c}-t)] + \mathrm{h.c.}\}\\
E_x &= i\mathrm{c}(\mu_0\hbar/2V_0)^{1/2}\sum_j \omega_j^{1/2}\{a_j \exp[i\omega_j(z/\mathrm{c}-t)] - \mathrm{h.c.}\}\\
B_y &= i(\mu_0\hbar/2V_0)^{1/2}\sum_j \omega_j^{1/2}\{a_j \exp[i\omega_j(z/\mathrm{c}-t)] - \mathrm{h.c.}\}\\
S_z &= -(\hbar\mathrm{c}/2V_0)\left\{\sum_j \omega_j^{1/2} a_j \exp[i\omega_j(z/\mathrm{c}-t)] - \mathrm{h.c.}\right\}^2\\
w &= -(\hbar/2V_0)\left\{\sum_j \omega_j^{1/2} a_j \exp[i\omega_j(z/\mathrm{c}-t)] - \mathrm{h.c.}\right\}^2
\end{aligned}
\tag{24}
$$

where the Hermite conjugate part (h.c.) contains the corresponding creation operators $a_j^\dagger$. In general, the sum goes over all circular frequencies ω_j. If these quantities are restricted to the values ω_1 and ω_2, then the expectation values can be evaluated with respect to a proper state of the phase

$$
\begin{aligned}
|\,\rangle = |\Theta_{m3}\rangle &= 3^{-1/2}\sum_{k=0}^{2}|3-k, k; m\rangle = 18^{-1/2}\{(a_1^\dagger)^3\\
&+ 3^{1/2}(a_1^\dagger)^2 a_2^\dagger \exp(2\pi i m/3) + 3^{1/2} a_1^\dagger (a_2^\dagger)^2 \exp(4\pi i m/3)\}|0,0\rangle
\end{aligned}
\tag{25}
$$

for a total occupation number $n = 3$. Especially for the component S_z of the Poynting vector and for the energy density w one gets the expectation values

$$
\begin{aligned}
\langle S_z\rangle = \mathrm{c}\langle w\rangle = (\hbar\mathrm{c}/2V_0)\Bigg\{ & 5\omega_1 + 3\omega_2\\
&+ \frac{4}{3}(2+3^{1/2})(\omega_1\omega_2)^{1/2}\cos\left[(\omega_2-\omega_1)\left(\frac{z}{\mathrm{c}} - t\right) + \frac{2\pi}{3}m\right]\Bigg\}.
\end{aligned}
\tag{26}
$$

The classical oscillator frequencies ω_1 and ω_2 do not appear as separated quantities. The time-dependence of the expectation value (26) is determined by the beat frequency $|\omega_2 - \omega_1|$. The sign of the difference $\omega_2 - \omega_1$ does not enter. The phases of the plane waves in the sums (24) vanish at $z = 0$ for $t = 0$. For the superposition (25) of quantum states this position is shifted by the length

$$z = 2\pi m c/\omega \text{ for } \omega = \omega_2 - \omega_1 > 0 \text{ and } m = 0, 1, 2.$$

If the difference between the circular frequencies is very small

$$\langle S_z \rangle = c\langle w \rangle = \frac{4c}{V_0}\hbar\omega_1 \left\{ 1 + \frac{1}{6}(2 + 3^{1/2}) \cos\left[\omega\left(\frac{z}{c} - t\right) + \frac{2\pi}{3}m\right]\right\}$$
$$\text{for } \omega \ll \omega_1, \quad (27)$$

then the ratio between the maximum and the minimum value of the energy flux density and of the energy density is given by the number

$$\frac{\langle w \rangle_{\max}}{\langle w \rangle_{\min}} = \frac{1}{13}(35 + 12 \cdot 3^{1/2}) = 4.2911.$$

This ratio should be measurable at positions whitin the wave-length

$$\lambda_b = \frac{2\pi c}{\omega} \quad (28)$$

of the beat. The time-dependence of the expectation values (26) or (27) is specific for the proper states (25) of the phase difference, whereas in a proper state

$$|2, 1\rangle = 2^{-1/2}(a_1^\dagger)^2 a_2^\dagger |0, 0\rangle$$

of the special occupation numbers n_1 and n_2 the beat vanishes

$$\langle 2, 1|S_z|2, 1\rangle = (\hbar c/2V_0)(5\omega_1 + 3\omega_2),$$

although the other contributions are the same as in the expressions (13) and (26).

Finally, the phase difference Θ_{mn} is a measurable quantity, because the maximum intensity can appear n times within the wave-length λ_b of the beat

$$z_{\max} - ct = (1 - m/n)\lambda_b. \quad (m = 1, 2, 3, \ldots, n) \quad (29)$$

The relation (21) can be interpreted in such a way that further maxima of the intensity can appear within the wave-length λ_b for an even occupation number n

$$z_{\max} - ct = (1/2 - m/n)\lambda_b. \quad (2m < n)$$

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Reference

[1] Sixia Yu, 1997, to be published.

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