Sitzungsber. Abt. II (2003) 212: 69–77

Sitzungsberichte
Mathematisch-naturwissenschaftliche Klasse Abt. II
Mathematische, Physikalische und Technische Wissenschaften

Continued Fractions with Increasing Digits

By

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(Vorgelegt in der Sitzung der math.-nat. Klasse am 9. Oktober 2003
durch das k. M. Fritz Schweiger)

Abstract

The most well known algorithm with increasing digits is Engel series. Recently it was shown that a certain continued fraction algorithm also produces increasing digits. Their stochastic behavior seems to be almost the same as is known for Engel series. In this note a whole class of algorithms with increasing digits is given. However, with proper choice of a parameter the stochastic properties are different from the already mentioned examples.

Mathematics Subject Classifications (2000): 11K55, 11A63, 28D99.
Key words: Metric theory, continued fractions, f-expansions.

1

The most well known example of an algorithm with increasing digits is Engel series. This algorithm is induced by the map $T\colon]0,1] \to]0,1]$ $T(x) = (k+1)x - 1$, $\frac{1}{k+1} < x \le \frac{1}{k}$, $k \ge 1$ (see e.g. Galambos, 1976; Perron, 1960).

Hartono et al. (2002) and Kraaikamp and Wu (2003) consider the continued fraction like map

$$S\colon]0,1] \to]0,1]$$
$$S(x) = \frac{1}{kx} - 1, \; \frac{1}{k+1} < x \le \frac{1}{k}.$$

Surprisingly, the ergodic behavior of both maps is very similar. For both algorithms one finds

(1) $\lim_{n\to\infty} \frac{\log k_n(x)}{n} = 1$ *a.e.*.
(2) The underlying maps T and S are ergodic with respect to Lebesgue measure.

One wonders if all maps with increasing digits behave in the same way. The aim of this note is to show that there are maps with increasing digits for which property (1) does not hold and property (2) is unlikely to be true.

Let

$$B(k) = \left]\frac{1}{k+1}, \frac{1}{k}\right], \quad k = 1, 2 \ldots \ .$$

Then we define the generalized continued fractions by the map

$$T_\epsilon(x) = \frac{-1 + (k+1)x}{1 + \epsilon - k\epsilon x}, \quad x \in B(k).$$

As usual we also consider the continuous extension

$$T_\epsilon^+\left(\frac{1}{k+1}\right) := \lim_{h\to 0+} T_\epsilon\left(\frac{1}{k+1} + h\right) = 0.$$

The parameter $\epsilon = \epsilon(k)$ should satisfy $\epsilon(k) + k + 1 > 0$.

Some elementary calculations show

(a) $T_\epsilon(\frac{1}{k}) = \frac{1}{k}$,
(b) $T'_\epsilon(x) = \frac{1+\epsilon+k}{(1+\epsilon-k\epsilon x)^2}$,
(c) $T'_\epsilon(\frac{1}{k}) = k + 1 + \epsilon$, $(T_\epsilon^+)'(\frac{1}{k+1}) := \lim_{h\to 0+} T'_\epsilon(\frac{1}{k+1} + h) = \frac{(k+1)^2}{k+1+\epsilon}$.

We distinguish three cases

(1) $-k-1 < \epsilon < 0$: Then the pole of T_ϵ satisfies

$$\xi = \frac{1+\epsilon}{k\epsilon} < \frac{1}{k+1}.$$

(1.1) $-k-1 < \epsilon < -k$: T_ϵ has a fixed point

$$\alpha(k) = -\frac{1}{\epsilon(k)} \quad \text{with} \quad \frac{1}{k+1} < \alpha < \frac{1}{k} \quad \text{and} \quad 0 < T'_\epsilon\left(\frac{1}{k}\right) < 1.$$

(1.2) $-k \le \epsilon < 0$: T_ϵ has no fixed point in the open interval

$$\left]\frac{1}{k+1}, \frac{1}{k}\right[\quad \text{and} \quad T'_\epsilon\left(\frac{1}{k}\right) \ge 1.$$

(2) $\epsilon = 0$: Then $T_0(x) = (k+1)x - 1$ is the map associated with Engel series.

(3) $\epsilon > 0$: Then

$$\xi = \frac{1+\epsilon}{k\epsilon} > \frac{1}{k}.$$

(3.1) $0 < \epsilon \leq k^2 + k$: T_ϵ has no point of intersection with the line

$$y = x - \frac{1}{k+1}$$

in $B(k)$ and

$$1 \leq (T_\epsilon^+)'\left(\frac{1}{k+1}\right).$$

(3.2) $k^2 + k < \epsilon$: T_ϵ has a point of intersection with the line

$$y = x - \frac{1}{k+1}$$

in $B(k)$, namely

$$\beta(k) = \frac{\epsilon - k}{k\epsilon} \quad \text{and} \quad 0 < (T_\epsilon^+)'\left(\frac{1}{k+1}\right) < 1.$$

As usual we define $k_s(x) = k$ if $T_\epsilon^{s-1}(x) \in B(k)$. Then the digits satisfy $k_1(x) \leq k_2(x) \leq k_3(x) \leq \ldots$.

The local inverse branches of T_ϵ are given as

$$V(k)x = \frac{1 + (1+\epsilon)x}{k + 1 + k\epsilon x}.$$

Therefore we find

$$V(k_1, \ldots, k_s) = \frac{A_s + B_s x}{C_s + D_s x}$$

where the numbers A_s, B_s, C_s, D_s satisfy the relation $(k_{s+1} = b)$

$$\begin{pmatrix} C_{s+1} & D_{s+1} \\ A_{s+1} & B_{s+1} \end{pmatrix} = \begin{pmatrix} C_s & D_s \\ A_s & B_s \end{pmatrix} \begin{pmatrix} b+1 & b\epsilon \\ 1 & 1+\epsilon \end{pmatrix}$$

Therefore

$$\lambda(B(k_1, \ldots, k_s)) = \frac{C_s B_s - A_s D_s}{C_s(C_s k_s + D_s)}.$$

A further calculation shows

$$\lambda(B(k_1,\dots,k_s,b)) = \frac{C_sB_s - A_sD_s}{(C_sb+D_s)(C_s(b+1)+D_s)}.$$

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We now suppose: $\epsilon(k) \geq (1+\gamma)(k^2+k)$ for some constant $\gamma > -1$.

Lemma. $D_s \geq k_s^2(1+\gamma)C_s$.

Proof. For $s=1$ we see $D_1 = k_1\epsilon \geq k_1^2(1+\gamma)C_1 = k_1^2(k_1+1)(1+\gamma)$. Then a calculation shows

$$\begin{aligned} D_{s+1} &= b\epsilon C_s + (1+\epsilon)D_s \\ &\geq b^2(b+1)(1+\gamma)C_s + b^2(1+\gamma)D_s \\ &= b^2(1+\gamma)C_{s+1}. \end{aligned}$$

Theorem. *For almost all x we find $k_{s+1} \geq k_s^2 + k_s$ for infinitely many values of s.*

Proof. A calculation shows

$$\begin{aligned} &\sum_{k_s \leq b < k_s(k_s+1)} \frac{\lambda(B(k_1,\dots,k_s,b))}{\lambda(B(k_1,\dots,k_s))} \\ &\quad = \sum_{k_s \leq b < k_s(k_s+1)} \frac{C_s(C_sk_s+D_s)}{(C_sb+D_s)(C_s(b+1)+D_s)} \\ &\quad = \frac{C_sk_s^2}{C_s(k_s^2+k_s)+D_s} \leq \frac{1}{2+\gamma} < 1. \end{aligned}$$

Remark. *Note that the condition $\epsilon \geq (1+\gamma)(k^2+k)$ for a constant $\gamma > -1$ includes all maps from case* (3.2) *and a range within case* (3.1). *It is very likely that these maps are not ergodic* (*compare known results on Sylvester series*).

3

We now suppose $-b-1 < \epsilon(b) \leq -b-\rho$ for a constant $0 < \rho < 1$. We introduce the sets

$$B(k^+) := \left]\alpha(k),\ \frac{1}{k}\right], \qquad B(k^-) := \left]\frac{1}{k+1},\ \alpha(k)\right].$$

Note that $TB(k^+) = B(k^+)$. Therefore if $(k_1^-, k_2^-, \dots, k_s^-, b^+)$ is an admissible block, then $k_s < b$.

Theorem. *The set* $E := \bigcup_{r=1}^{\infty} B(r^+)$ *is absorbing.*

Proof. Calculations show

$$\lambda(B(k_1^-,\ldots,k_s^-)) = \int_0^{\alpha(k_s)} \frac{C_sB_s - A_sD_s}{(C_s + D_sx)^2}dx = \frac{(C_sB_s - A_sD_s)\alpha(k_s)}{C_s(C_s + D_s\alpha(k_s))}$$
$$= \frac{(C_sB_s - A_sD_s)}{C_s(-C_s\epsilon(k_s) + D_s)},$$
$$\lambda(B(k_1^-,\ldots,k_s^-,b^+)) = \int_{\alpha(b)}^{\frac{1}{b}} \frac{C_sB_s - A_sD_s}{(C_s + D_sx)^2}dx$$
$$= \frac{(C_sB_s - A_sD_s)(b+\epsilon(b))}{(C_s\epsilon(b) - D_s)(C_sb + D_s)}.$$

We first observe

$$\frac{b+\epsilon(b)}{C_s\epsilon(b) - D_s} = \frac{-b + |\epsilon(b)|}{C_s|\epsilon(b)| + D_s} \geq \frac{\rho}{C_s(b+1) + D_s}.$$

Hence

$$\sum_{b=k_s+1}^{\infty} \lambda(B(k_1^-,\ldots,k_s^-,b^+)) \geq \frac{\rho(C_sB_s - A_sD_s)}{C_s(C_s(k_s+1) + D_s)}.$$

Next we estimate the ratio

$$\frac{-C_s\epsilon(k_s) + D_s}{C_s(k_s+1) + D_s} = \frac{C_s|\epsilon(k_s)| + D_s}{C_s(k_s+1) + D_s} \geq \frac{C_s(k_s+\rho) + D_s}{C_s(k_s+1) + D_s} \geq \rho.$$

Since $C_sk_s + D_s > 0$, this is true.

Therefore

$$\sum_{k_{s+1}=k_s}^{\infty} \lambda(B(k_1^-,\ldots,k_s^-,k_{s+1}^-)) \leq (1-\rho^2)\lambda(B(k_1^-,\ldots,k_s^-)).$$

This shows that

$$\lambda\left(\bigcap_{s=1}^{\infty} \bigcup_{k_1 \leq \cdots \leq k_s} B(k_1^-,\ldots,k_s^-)\right) = 0.$$

4

We now consider maps for which we can show that they are ergodic.

Theorem. *Suppose there is a constant* $0 < \kappa \leq 1$ *such that* $|\epsilon(b)| \leq b^{1-\kappa}$, *then* T_ϵ *is ergodic with respect to Lebesgue measure.*

Proof. We follow the ideas outlined in HARTONO et al. (2002) (see also THALER, 1979).

Observe that

$$\omega(k_1, \ldots, k_s; x) = \frac{B_s C_s - A_s D_s}{(C_s + D_s x)^2}.$$

Furthermore since $-k \le \epsilon(k)$ either $T_\epsilon^s x = \frac{1}{b}$ for some s or $\lim_{s\to\infty} k_s(x) = \infty$.

We first show

$$\frac{1}{2} \le \frac{k_s C_s}{k_s C_s + D_s} \le 2$$

for all $k_s \ge K(\kappa)$.

(a) If $\epsilon \ge 0$ then $C_s \ge 0$ and $D_s \ge 0$. Therefore

$$\frac{k_s C_s}{k_s C_s + D_s} \le 1.$$

We claim $D_s \le k_s C_s$. This is true for $s = 1$.

Furthermore

$$\begin{aligned} D_{s+1} &= b\epsilon C_s + \epsilon D_s + D_s \le b^2 C_s + bD_s + k_s C_s \\ &\le b^2 C_s + bD_s + bC_s = bC_{s+1}. \end{aligned}$$

(b) Now let $-b^{1-\kappa} \le \epsilon < 0$. We prove by induction the following three inequalities

$$0 < C_s, \qquad D_s < 0, \qquad C_s + k_s^{-1+\kappa} D_s > 0.$$

Note that for any w with $0 \le w < k_s^{-1+\kappa}$ we get $C_s + wD_s < 0$. Since $D_1 = b\epsilon$ and $C_1 = b + 1$ we see that

$$C_1 + D_1 b^{-1+\kappa} = b + 1 + \epsilon b^\kappa \ge b + 1 - b > 0.$$

We further calculate

$$C_{s+1} = (b+1)C_s + D_s = (b+1)\left(C_s + \frac{1}{b+1} D_s\right) > 0$$

and

$$D_{s+1} = b\epsilon C_s + (\epsilon + 1) D_s = \epsilon b \left(C_s + \frac{1}{b} D_s\right) + D_s < 0,$$

$$\begin{aligned} C_{s+1} + b^{-1+\kappa} D_{s+1} &= C_s(b + 1 + b^\kappa \epsilon) + D_s(1 + \epsilon b^{-1+\kappa} + b^{-1+\kappa}) \\ &= (C_s + b^{-1+\kappa} D_s) + (C_s b + D_s)(1 + b^{-1+\kappa}\epsilon). \end{aligned}$$

Since $b \geq k_s$ and $D_s < 0$ we get

$$C_s + b^{-1+\kappa} D_s \geq C_s + k_s^{-1+\kappa} D_s > 0$$

and

$$C_s b + D_s = b\left(C_s + \frac{1}{b} D_s\right) > 0.$$

Furthermore $1 + b^{-1+\kappa}\epsilon \geq 0$. Hence we obtain $C_{s+1} + b^{-1+\kappa} D_{s+1} > 0$. Then we finally obtain

$$\frac{k_s C_s}{k_s C_s + D_s} \leq \frac{k_s^{\kappa}}{k_s^{\kappa} - 1} \leq 2 \quad \text{for} \quad k_s \geq K(\kappa).$$

We could also replace the constant 2 on the right-hand side by $c(\kappa)$, say and we obtain

$$\frac{k_s C_s}{k_s C_s + D_s} \leq \frac{k_s^{\kappa}}{k_s^{\kappa} - 1} \leq c(\kappa)$$

for $k_s \geq 2$. It is not possible to include $k_s = 1$ in such an estimate as the case $\epsilon(1) = -1$ shows.

Now let A be an invariant set, i.e. $T^{-1}A = A$. We define

$$d(b) := b \int_0^{\frac{1}{b}} c_A(t)dt$$

and

$$\delta(b_1,\ldots,b_s) := \frac{\lambda(A \cap B(b_1,\ldots,b_s))}{\lambda(B(b_1,\ldots,b_s))} = \frac{\lambda(T^{-s}A \cap B(b_1,\ldots,b_s))}{\lambda(B(b_1,\ldots,b_s))}$$
$$= \left(\int_0^{\frac{1}{b_s}} c_A(t)\omega(b_1,\ldots,b_s;\, t)dt\right)\left(\int_0^{\frac{1}{b_s}} \omega(b_1,\ldots,b_s;\, t)dt\right)^{-1}.$$

Then

$$\frac{1}{16} d(b_s) \leq \delta(b_1,\ldots,b_s) \leq 16 d(b_s) \quad \text{and}$$
$$d(b) - d(b+1) = \frac{1}{b+1}(\delta(b) - d(b+1)).$$

For $s = 1$ we need a more careful estimate.

If $\epsilon > 0$ then

$$\frac{1}{b+1+\epsilon} \leq \omega(b;t) \leq \frac{b+1+\epsilon}{(b+1)^2}.$$

If $\epsilon < 0$ then

$$\frac{b+1+\epsilon}{(b+1)^2} \leq \omega(b;t) \leq \frac{1}{b+1+\epsilon}.$$

We calculate

$$\delta(b) = b(b+1)\int_{\frac{1}{b+1}}^{\frac{1}{b}} c_A(t)dt$$

$$= b(b+1)\int_0^{\frac{1}{b}} \omega(b;t)c_A(t)dt.$$

If $\epsilon > 0$ then

$$\delta(b) \geq \frac{b+1}{b+1+\epsilon} d(b).$$

Since $bd(b+1) = (b+1)d(b) - \delta(b)$ we obtain

$$d(b+1) \leq \left(1 + \frac{\epsilon}{b^2+b+b\epsilon}\right) d(b).$$

If $\epsilon < 0$ then

$$\delta(b) \geq \frac{b+1+\epsilon}{b+1} d(b)$$

and we obtain

$$d(b+1) \leq \left(1 - \frac{\epsilon}{b^2+b}\right) d(b).$$

Since $|\epsilon| \leq b^{1-\kappa}$ the products $\prod_{b=1}^{\infty}(1+\frac{\epsilon}{b^2+b+b\epsilon})$ and $\prod_{b=1}^{\infty}(1-\frac{\epsilon}{b^2+b})$ both are convergent. Therefore there is a constant $\gamma \geq 0$ such that $d(c) \leq \gamma d(b)$ for all $c \geq b$.

The martingale theorem now shows that

$$\lim_{s\to\infty}\delta(b_1(x),\ldots,b_s(x)) = c_A(x)$$

almost everywhere.

Since $\lim_{s\to\infty} b_s(x) = \infty$ almost everywhere for almost all points x, y and numbers $r \geq 1$ there is a number $s \geq r$ such that

$$\delta(b_1(x),\ldots,b_s(x)) \leq 256\gamma\delta(b_1(y),\ldots,b_r(y)).$$

Therefore, if $\lambda(A) < 1$ we immediately get $\lambda(A) = 0$.

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