Anzeiger Abt. II (2003) 139: 17–19

Anzeiger

Mathematisch-naturwissenschaftliche Klasse Abt. II
Mathematische, Physikalische und Technische Wissenschaften

On the Distribution of the Number of Vertices of a Random Polygon

By

Christian Buchta

(Vorgelegt in der Sitzung der math.-nat. Klasse am 11. Dezember 2003
durch das w. M. August Florian)

Abstract

Assume that n points $P_1, \ldots, P_n$ are distributed independently and uniformly in the triangle with vertices (0, 1), (0, 0), and (1, 0). Consider the convex hull of (0, 1), $P_1, \ldots, P_n$, and (1, 0). Denote by N_n the number of those points $P_1, \ldots, P_n$ which are vertices. Let $p_k^{(n)} (k = 1, \ldots, n)$ be the probability that $N_n = k$. Bárány, Rote, Steiger, and Zhang [1] proved that $p_n^{(n)} = 2^n/[n!(n+1)!]$. We derive for $k = 1, \ldots, n-1$ the values of $p_k^{(n)}$ and thus obtain the exact distribution of N_n. Knowing this distribution provides the key to the answer of some long-standing questions in geometrical probability, e.g., to the distribution of the number of vertices of the convex hull of n points distributed independently and uniformly in a convex polygon.

Mathematics Subject Classification (2000): 52A22, 60D05.
Key words: Convex hull, random points, random polygons.

*

Theorem 1. *For $n \in \mathbb{N}$ and $k = 1, \ldots, n$ the probabilities $p_k^{(n)}$ are given by*

$$p_k^{(n)} = \frac{2}{n(n+1)} \sum_{j=k-1}^{n-1} (n-j) p_{k-1}^{(j)},$$

with $p_0^{(0)} = 1$ *and* $p_0^{(j)} = 0$ *for* $j \in \mathbb{N}$. *Alternatively, these probabilities are given by*

$$p_k^{(n)} = 2^k \sum \frac{i_1 \cdots i_k}{i_1(i_1+1)(i_1+i_2)(i_1+i_2+1)\cdots(i_1+\cdots+i_k)(i_1+\cdots+i_k+1)},$$

where the sum is taken over all $i_1, \ldots, i_k \in \mathbb{N}$ *such that* $i_1 + \cdots + i_k = n$.

The proof of Theorem 1 will be published in a forthcoming paper, which will also describe how the distribution of the number of vertices of the convex hull of n points distributed independently and uniformly in a convex polygon arises in terms of the probabilities $p_k^{(n)}$. Here we will only state the following consequence of Theorem 1:

Theorem 2. *The expected value and the variance of the random variable* N_n *are given by*

$$EN_n = \frac{1}{3}\left(2\sum_{k=1}^{n}\frac{1}{k} + 1\right),$$

$$\operatorname{var} N_n = \frac{1}{27}\left(10\sum_{k=1}^{n}\frac{1}{k} + 12\sum_{k=1}^{n}\frac{1}{k^2} - 28 + \frac{12}{n+1}\right).$$

The asymptotic version of the first formula in Theorem 2, i.e. $EN_n \sim \frac{2}{3}\log n$ as n tends to infinity, is a classical result due to Rényi and Sulanke [4]. The asymptotic version of the second formula, i.e. $\operatorname{var} N_n \sim \frac{10}{27}\log n$ as n tends to infinity, is due to Groeneboom [3]. It was obtained by approximating the process of vertices of the convex hull of a uniform sample by the process of extreme points of a realization of a Poisson point process: The "left-lower boundary" of the convex hull of a uniform sample of size n from the interior of the square $[0, \sqrt{n}]^2$ is associated with the "left-lower boundary" of the convex hull of a realization of a Poisson point process on $\mathbb{R}_+^2$ with intensity Lebesgue measure. For comments on Groeneboom's paper see [2].

References

[1] Bárány, I., Rote, G., Steiger, W., Zhang, C.-H. (2000) A central limit theorem for convex chains in the square. Discrete Comput. Geom. **23**: 35–50

[2] Buchta, C. (to appear) An identity relating moments of functionals of convex hulls. Discrete Comput. Geom.

[3] GROENEBOOM, P. (1988) Limit theorems for convex hulls. Probab. Theory Relat. Fields **79**: 327–368
[4] RÉNYI, A., SULANKE, R. (1963) Über die konvexe Hülle von n zufällig gewählten Punkten. Z. Wahrscheinlichkeitsth. Verw. Geb. **2**: 75–84

Author's address: Prof. Dr. Christian Buchta, Institut für Mathematik, Universität Salzburg, Hellbrunner Straße 34, 5020 Salzburg, Austria.