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Laplace Problems for Regular Lattices with an Even Number of Different Obstacles

By

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Abstract

In this paper we consider some regular lattices with fundamental cell with a even number of obstacles. In particular we obtain the Laplace probability.

Key words: Geometric probability, stochastic geometry, random sets, random convex sets and integral geometry.

1. Section

Let $\Re_1(a, b, c)$ be the regular lattice with fundamental cell is as in Fig. 1.

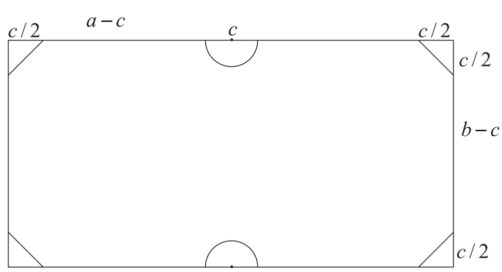

Fig. 1

Denoting with $C_0^{(1)}$ the fundamental cell of this lattice, we have:

$$\text{area}\, C_0^{(1)} = 2ab - \frac{(2+\pi)c^2}{4}.$$

The cell $C_0^{(1)}$ has six obstacles that are quarter-squares with diagonal of length c with $c < \min(a, b)$ and semi circles with diameter c.

Considering a segment s of random position and of constant length l with $c < l < \min(a, b)$, we want compute the probability that this segment intersects a side of lattice; obviously this probability is equal to probability $P_{\text{int}}^{(1)}$ that the segment s intersects the bounderay of the fundamental cell.

The position of the segment s is determinated by the middle point O and by the angle φ that the segment forms with the axis x. We consider the limit positions of the segment s that corrisponde at angle φ and let $\widehat{C}_0^{(1)}(\varphi)$ the determinated figure from these positions (Fig. 2):

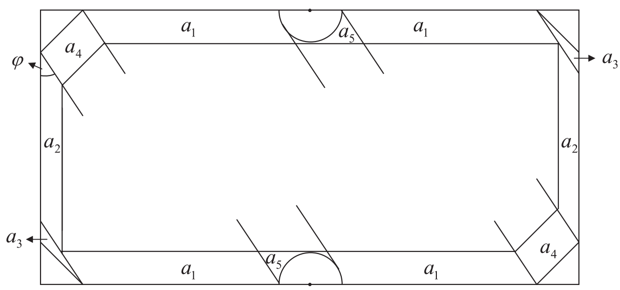

Fig. 2

From this figure we can write:

$$\begin{aligned}\text{area}\, \widehat{C}_0^{(1)}(\varphi) = \text{area}\, C_0^{(1)} &\\ - [4\text{area}\, a_1(\varphi) + 2\text{area}\, a_2(\varphi) + 2\text{area}\, a_3(\varphi) &\\ + 2\text{area}\, a_4(\varphi) + 2\text{area}\, a_5(\varphi)]. &\qquad (1)\end{aligned}$$

Considering some results that we have obtained in a previous paper [1], follow that:

$$\text{area}\, a_1(\varphi) = \frac{(a-c)l}{2}\cos\varphi, \quad \text{area}[a_2(\varphi) + a_3(\varphi)] = \frac{(b-c)l}{2}\sin\varphi,$$

$$\text{area}\, a_4(\varphi) = \frac{cl}{4}(\sin\varphi + \cos\varphi). \qquad (2)$$

But from Fig. 3:

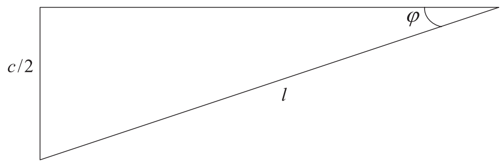

Fig. 3

we have $c = 2l \sin \varphi$, hence

$$\text{area}\, a_4(\varphi) = \frac{cl}{4}(\sin\varphi + 2\cos\varphi) - \frac{l^2}{4}\sin 2\varphi.$$

In order to compute area $a_5(\varphi)$, we consider the figure

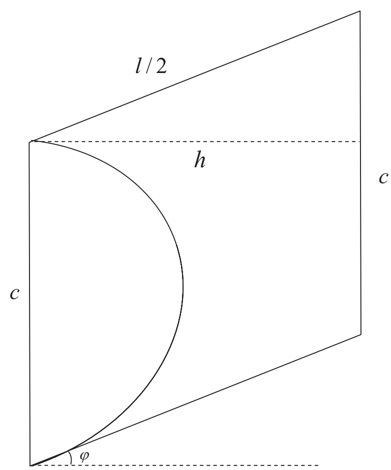

Fig. 4

From here follows $h = \frac{l}{2}\cos\varphi$, hence

$$\text{area}\, a_5(\varphi) = \frac{cl}{2}\cos\varphi - \frac{\pi c^2}{8} \quad (3)$$

Replacing in the formula (1) le expressions (2), (3) and (4) we obtain

$$\text{area}\, \widehat{C}_0^{(1)}(\varphi) = \text{area}\, C_0^{(1)} - \left[2al\cos\varphi + (b - \frac{c}{2})l\sin\varphi - \frac{l^2}{2}\sin 2\varphi - \frac{\pi c^2}{4}\right]. \quad (4)$$

Denoting with M_1 the set of segments s whose the middle point are in $C_0^{(1)}$ and N_1 the set of segments s completely contained in $C_0^{(1)}$, we have that:

$$P_{\text{int}}^{(1)} = 1 - \frac{\mu(N_1)}{\mu(M_1)}, \tag{5}$$

where μ is the Lebesgue measure in Euclidean plane [3].

In order to compute the measures $\mu(M_1)$ and $\mu(N_1)$ we use the Poincaré kinematic measure [2]

$$dK = dx \wedge dy \wedge d\varphi,$$

where x, y are the coordinates of O and φ the defined angle.

Since $\varphi \in \left[0, \frac{\pi}{2}\right]$, we have:

$$\mu(M_1) = \int_0^{\frac{\pi}{2}} d\varphi \iint_{\{(x,y)\in C_0^{(1)}\}} dxdy = \frac{\pi}{2}\text{area}\, C_0^{(1)} = \frac{\pi}{2}\left[2ab - \frac{(2+\pi)c^2}{4}\right]. \tag{6}$$

and, considering the (5)

$$\mu(N_1) = \int_0^{\frac{\pi}{2}} d\varphi \iint_{\left\{(x,y)\in \hat{C}_0^{(1)}(\varphi)\right\}} dxdy = \int_0^{\frac{\pi}{2}} \text{area}\, \widehat{C}_0^{(1)}(\varphi) d\varphi$$

$$= \frac{\pi}{2}\left[2ab - \frac{(2+\pi)c^2}{4}\right] - \int_0^{\frac{\pi}{2}} \left[2al\cos\varphi + \left(b - \frac{c}{2}\right) l\sin\varphi - \frac{l^2}{2}\sin 2\varphi - \frac{\pi c^2}{4}\right] d\varphi$$

$$= \frac{\pi}{2}\left[2ab - \frac{(2+\pi)c^2}{4}\right] - \left[2al\sin\varphi - \left(b - \frac{c}{2}\right) l\cos\varphi + \frac{l^2}{4}\cos 2\varphi - \frac{\pi c^2}{4}\varphi\right]$$

$$= \frac{\pi}{2}\left[2ab - \frac{(2+\pi)c^2}{4}\right] - \left[\left(2a + b - \frac{c}{2}\right) l - \frac{l^2}{2} - \frac{\pi^2 c^2}{8}\right]. \tag{7}$$

The formulas (6), (7) and (8) give us that:

$$P_{\text{int}}^{(1)} = \frac{2\left(2a + b - \frac{c}{2}\right) l - l^2 - \frac{\pi^2 c^2}{4}}{\pi\left[2ab - \frac{(2+\pi)c^2}{4}\right]}. \tag{8}$$

When $c \to 0$, the obstacles becames points and the fundamental cell becames a rectangle with side $2a$ and b. In this case the probability (9) becomes the Laplace probability:

$$P = \frac{2(2a + b)l - l^2}{2\pi ab}.$$

2. Section

Let $\Re_2(a,b,c,n)$ be the regular lattice with fundamental cell $C_0^{(2)}$ a rectangle with side $(n+1)a$ e b and with $4(n+1)$ obstacles: $2n$ semi circles with the radius $\frac{c}{2}$, 2 quarter-circles with the same radius, $2n$ semi square and 2 quarter-square with the diagonal c with $c<\min[(n+1)\,a,\ b]$ (Fig. 5):

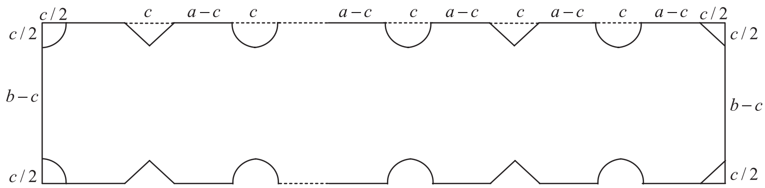

Fig. 5

We have:

$$\text{area}\, C_0^{(2)} = (n+1)ab - \frac{c^2}{4}\left(n+1-\frac{4n+1}{4}\pi\right).$$

In the same way of the Section 1, considering a segment s of random position and of constant length l with $c<l<\min[(n+1)\,a,\ b]$.

We want compute the probability that this segment intersects a side of lattice, obviously this probability is equal to probability $P_{\text{int}}^{(2)}$ that the segment s intersects the bounderay of the fundamental cell.

The position of the segment s is determinated by the middle point O and by the angle φ that the segment forms with the axis x. We consider the limit positions of the segment s that corrisponde at angle φ and let $\widehat{C}_0^{(2)}(\varphi)$ the determinated figure from these positions (Fig. 6):

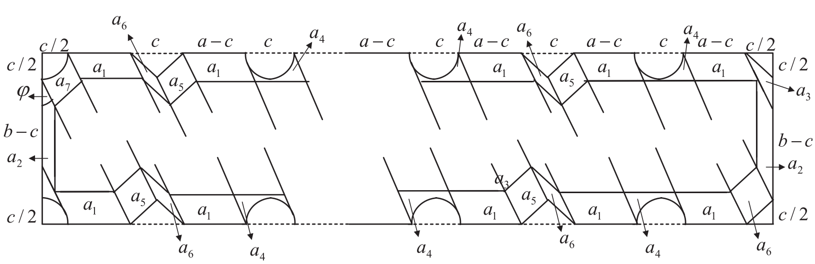

Fig. 6

From this figure we can write:

$$\begin{aligned}\text{area}\,\widehat{C}_0^{(2)}(\varphi) = {} & \text{area}\,C_0^{(2)} - 2(n+1)\text{area}\,a_1(\varphi) + 2\text{area}\,a_2(\varphi) \\ & + \text{area}\,a_3(\varphi) + 2n\text{area}\,a_4(\varphi) + n\text{area}\,a_5(\varphi) + n\text{area}\,a_6(\varphi) \\ & + \text{area}\,a_7(\varphi) + \text{area}\,a_8(\varphi) + \text{area}\,a_9(\varphi). \qquad (9)\end{aligned}$$

Considering of some results that we have obtained in a previous paper [1], we have:

$$\text{area}\,a_1(\varphi) = \frac{(a-c)l}{2}\cos\varphi, \quad \text{area}\,a_2(\varphi) = \left(b - \frac{c}{2} - l\cos\varphi\right)\frac{l}{2}\sin\varphi,$$

$$\text{area}\,a_3(\varphi) = \frac{cl}{2}\cos\varphi - \frac{c^2}{8},$$

and

$$\begin{aligned}\text{area}\,a_7(\varphi) &= \frac{cl}{4}(\sin\varphi + \cos\varphi) - \frac{\pi c^2}{8} + \frac{c^2}{8}, \\ \text{area}\,a_8(\varphi) &= \frac{cl}{4}\cos\varphi - \frac{\pi c^2}{16}, \qquad \text{area}\,a_9(\varphi) = \frac{cl}{4}(\sin\varphi + \cos\varphi). \qquad (10)\end{aligned}$$

In order to compute area $a_4(\varphi)$ we consider Fig. 7:

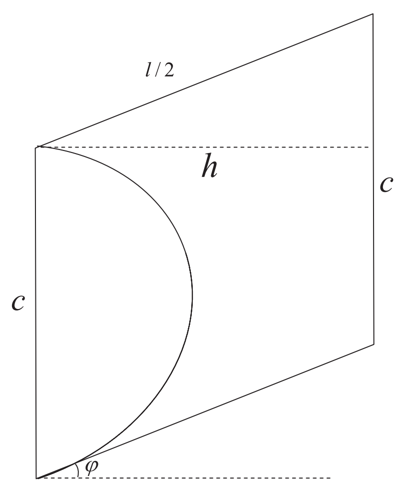

Fig. 7

From here follows $h = \frac{l}{2}\cos\varphi$, hence

$$\text{area}\, a_4(\varphi) = \frac{cl}{2}\cos\varphi - \frac{\pi c^2}{8} \tag{11}$$

From Fig. 8:

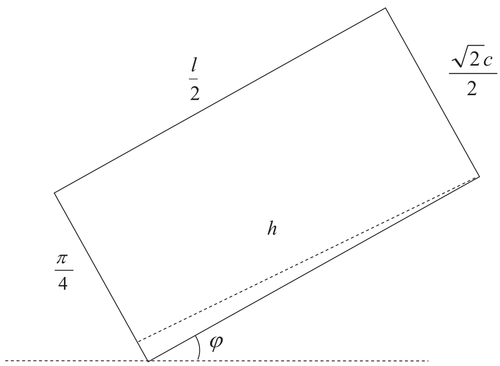

Fig. 8

we have

$$h = \frac{l}{2}\sin\left(\frac{\pi}{4} + \varphi\right) = \frac{l\sqrt{2}}{4}(\sin\varphi + \cos\varphi)$$

then

$$\text{area}\, a_5(\varphi) = \frac{cl}{4}(\sin\varphi + \cos\varphi) \tag{12}$$

In the end, from Fig. 9:

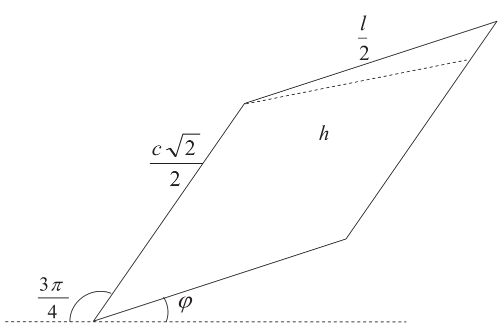

Fig. 9

follows:

$$h = \frac{l}{2}\sin\left(\frac{\pi}{4} - \varphi\right) = \frac{l\sqrt{2}}{4}(\cos\varphi - \sin\varphi)$$

then

$$\text{area}\, a_6(\varphi) = \frac{cl}{4}(\cos\varphi - \sin\varphi) \tag{13}$$

Replacing in the (10) the expression (11), (12), (13) and (14) we obtain:

$$\begin{aligned}\text{area}\, \hat{C}_0^{(2)}(\varphi) = \text{area}\, C_0^{(2)} - \Bigg\{ \left[(n+1)a + \frac{2n+1}{4}c\right] l\cos\varphi + bl\sin\varphi \\ - \frac{l^2}{2}\sin 2\varphi - \frac{(4n+3)\pi c^2}{16}\Bigg\}\end{aligned} \tag{14}$$

Denoting with M_2 the set of segments s whose the middle point are in $C_0^{(2)}$ and N_2 the set of segments s completely contained in $C_0^{(2)}$, we have, as above, that:

$$P_{\text{int}}^{(2)} = 1 - \frac{\mu(N_2)}{\mu(M_2)}. \tag{15}$$

Since $\varphi \in \left[0, \frac{\pi}{2}\right]$, we have:

$$\begin{aligned}\mu(M_2) &= \int_0^{\frac{\pi}{2}} d\varphi \iint_{\{(x,y)\in C_0^{(2)}\}} dxdy = \frac{\pi}{2}\,\text{area}\, C_0^{(2)} \\ &= \frac{\pi}{2}\left[(n+1)ab - \frac{c^2}{4}\left(n + 1 - \frac{4n+1}{4}\pi\right)\right]\end{aligned}$$

and considering the (15),

$$\mu(N_2)=\int_0^{\frac{\pi}{2}} d\varphi \iint_{\{(x,y)\in\hat{C}_0^{(2)}(\varphi)\}} dxdy=\int_0^{\frac{\pi}{2}}\text{area } C_0^{(2)}(\varphi)d\varphi$$

$$=\frac{\pi}{2}\text{area } C_0^{(2)}-\int_0^{\frac{\pi}{2}}\Bigg\{\left[(n+1)a+\frac{2n+1}{4}c\right]l\cos\varphi+bl\sin\varphi -\frac{l^2}{2}\sin 2\varphi-\frac{(4n+3)\pi c^2}{16}\Bigg\}d\varphi$$

$$=\text{area } C_0^{(2)}-\left\{\left[(n+1)a+b+\frac{2n+1}{4}c\right]l-\frac{l^2}{2}-\frac{(4n+3)\pi^2c^2}{32}\right\}. \quad (16)$$

The formulas (16), (17) and (18) give us that:

$$P_{\text{int}}^{(2)}=\frac{2\left[(n+1)a+b+\frac{2n+1}{4}c\right]l-l^2-\frac{(4n+3)\pi^2c^2}{16}}{\pi\left[(n+1)ab-\frac{c^2}{4}\left(n+1-\frac{4n+1}{4}\pi\right)\right]}. \quad (17)$$

When $c\to 0$, the obstacles becames points and the fundamental cell becames a rectangle with side $(n+1)a$ and b. In this case the probability (18) becomes the Laplace probability:

$$P=\frac{2[(n+1)a+b]l-l^2}{(n+1)\pi ab}.$$

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