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RESEARCH ARTICLE

Delayed reproduction has unexpected effects on population growth and structure

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ABSTRACT It is widely accepted that delayed reproduction reduces the population growth rate, with associated effects on population structure and size. Policies (e.g., "later, longer, fewer") have been based on this conclusion. However, it is rarely noted that the negative effect of reproductive delay on population growth applies to populations with positive growth rates. Many countries now experience below-replacement fertility levels and growth rates that would lead to population decline. In such populations, delayed reproduction increases, rather than decreases, population growth. This paper calculates the effects of delayed reproduction on the population growth rate, the population age distribution, and the equilibria of stationary-through-immigration populations. It does so for reproduction measured by age-specific fertility and by the parity transition matrix. In populations with below-replacement fertility, delayed reproduction leads to higher, not lower, population growth; to younger, not older, populations; and to larger, not smaller, equilibria. Examples are presented using age-specific rates for Japan and age × parity-specific rates for Slovakia; in both cases over a demographic transition from positive to negative growth.

KEYWORDS Population growth rate • Population aging • Delayed age-specific fertility • Delayed parity progression

Introduction

Recent studies of delayed reproduction, including papers in this issue, often focus on causes and mechanisms, on individual choices, and on technological developments (e.g., advances in medically assisted reproduction and egg freezing) that are providing new choices that were not previously available. However, delayed fertility also has effects at the population level – on growth rates, age distributions, and population size – that deserve consideration.

A majority of the world's population now lives in countries with below-replacement fertility levels.[2] The resulting concerns about population decline and ageing (surveyed by Skirbekk, 2022) have spilled from the pages of academic demography journals to become a major cultural and political issue and sources of inspiration for pro-natalist

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2 Our World in Data, https://ourworldindata.org/global-decline-fertility-rate

movements. This paper aims to present some possibly surprising results about the effects of delayed reproduction on population growth and structure in the current demographic situation of declining populations. We will see that delayed reproduction, all else being equal, may increase, rather than decrease population growth.

The effects of delayed reproduction have long been of interest to researchers. The earliest study on this topic seems to be that of Dublin and Lotka (1925). Analysing U.S. life tables from 1920, they found that a fixed postponement of reproduction by five years reduced the intrinsic rate of increase from $r = 0.0055$ to $r = 0.0036$. Coale (1956) and Coale and Tye (1961) also concluded that delayed fertility, modelled as a shift in the fertility function along the age axis, would reduce the population growth rate and lead to older populations. Coale and Tye (1961) did note in passing that if r were “sufficiently negative” the effect of delayed reproduction might be positive, but they did not explore this possibility further.

Population biologists and evolutionary demographers also addressed the question of the timing of reproduction, perhaps because of the much greater range of life histories, and thus of development patterns, found across the plant and animal kingdoms.[3] Early contributions include those of Cole (1954, 1965), Smith (1954), and Slobodkin (1961). The focus of this research was on the ways in which earlier reproduction could increase the population growth rate, and thus increase fitness. One of the most influential studies was that of Lewontin (1965), which was presented in a symposium on the genetics of colonising species. A colonising species, in this usage, might be a new arrival on a newly created island (MacArthur and Wilson, 1967) or a weed taking root in a newly ploughed field. In such cases, natural selection favours rapid population growth to take advantage of the situation.

Lewontin approached the problem by creating an idealised triangular net maternity function and shifting that function along the age axis in various ways. This allowed him to calculate the response of the intrinsic rate of increase r to changes in the timing of reproduction. His conclusion, supported by empirical examples of insect populations was that “the general point holds true that small absolute changes in developmental rates of the order of 10% are roughly equivalent to large increases in fertility of the order of 100%” (Lewontin, 1965, p. 85).

Caswell and Hastings (1980) were the first to use perturbation (i.e., sensitivity) analysis to explore the effects of reproductive timing on the population growth rate. They focused on *advanced* reproduction by shifting fertility one or more years earlier. In that case, if f_x is fertility at age x, then

$$\Delta f_x = f_{x+1} - f_x. \tag{1}$$

They calculated the marginal effect on the population growth rate λ as

$$\Delta\lambda = \sum_x \frac{\partial \lambda}{\partial f_x} \Delta f_x, \tag{2}$$

where λ is related to the continuous-time intrinsic rate of increase r by $r = \log \lambda$. For the most part their results echoed Lewontin’s, showing that earlier reproduction had a strongly

3 From aphids that are born already pregnant with the next generation (Gould, 1977) to bamboos that wait for a century for their one and only bout of reproduction (Janzen, 1976).

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positive effect on population growth, and that delayed reproduction would reduce population growth.

The conclusion that delayed reproduction would reduce the population growth rate became such a widely held belief that it became a basis for the “later, longer, fewer” birth planning campaign in China. Introduced in the early 1970s, this policy aimed to slow population growth by mandating later marriage (age 25 for women, age 27 or 28 for men) and longer birth intervals of at least four years (Whyte et al., 2015; Zhang, 2017). It was replaced by the one-child policy in 1979. The “later, longer, fewer” program was an explicit attempt to enforce delayed reproduction precisely because of its negative effects on population growth.

However, Caswell and Hastings (1980) had noted that in equilibrium or declining populations, delayed reproduction could increase the population growth rate. They speculated about the evolutionary implications of this observation, but did not pursue the issue further.

The current trend of declining fertility, the prevalence of below-replacement demographic rates, and the resulting concern about depopulation and population ageing suggest that it is time to re-examine the effects of delayed reproduction on population growth and structure. This paper brings to bear a much more powerful set of perturbation analyses to examine the effects of delayed reproduction.

Notation

In this paper, matrices are written as upper case bold characters (e.g., $\mathbf{U}$) and vectors as lower case bold characters (e.g., $\mathbf{a}$). Vectors are column vectors by default; $\mathbf{x}^\top$ is the transpose of $\mathbf{x}$. The ith unit vector (a vector with a one in the ith location and zeros elsewhere) is $\mathbf{e}_i$. The vector $\mathbf{1}$ is a vector of ones, and the matrix $\mathbf{I}$ is the identity matrix. When necessary, subscripts are used to denote the size of a vector or matrix; e.g., $\mathbf{I}_\omega$ is an identity matrix of size $\omega \times \omega$.

The symbol $\otimes$ denotes the Kronecker product. The vec operator stacks the columns of a $m \times n$ matrix into a $mn \times 1$ column vector. The notation $\|\mathbf{x}\|$ denotes the 1-norm of $\mathbf{x}$, i.e., the sum of the absolute values of the entries. MATLAB notation will be used to refer to rows and columns of a matrix, e.g., $\mathbf{F}(i,)$ and $\mathbf{F}(:,j)$ refer to the ith row and jth column of the matrix $\mathbf{F}$, respectively.

Sensitivity analysis and marginal effects

The effects of changes in the reproductive timing are obtained by calculating the derivative of some demographic outcome with respect to changes in some aspect of the fertility schedule. These derivatives are calculated using matrix calculus. This technique has been used in a long series of demographic papers published over the last 15 years (e.g., Caswell, 2008, 2010; Engelman et al., 2014; Van Raalte and Caswell, 2013). The reader is assumed to have some familiarity with the approach. A complete mathematical presentation can be found in Magnus and Neudecker (1988), and a more accessible treatment can be found in Abadir and

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Magnus (2005). There is also a book (Caswell, 2019) containing an elementary introduction to matrix calculus and presenting many demographic applications that is available for free download. The appendix of this paper gives some basic facts.

Briefly, if $\mathbf{y}$ and $\mathbf{x}$ are vectors, the derivative of $\mathbf{y}$ with respect to $\mathbf{x}$ must include the derivatives of all entries of $\mathbf{y}$ with respect to each of the entries of $\mathbf{x}$. If $\mathbf{y}$ is of length n and x is of length m, the derivative is the $n \times$ m matrix

$$\frac{d\mathbf{y}}{d\mathbf{x}^{\mathsf{T}}} = \left(\frac{dy_i}{dx_j}\right). \tag{3}$$

The derivative of a matrix $\mathbf{Y}$ with respect to a matrix $\mathbf{X}$ has even more entries. It is obtained by first applying the vec operator to transform the matrices to vectors, leading to

$$\frac{d\text{vec}\mathbf{Y}}{d\text{vec}^{\mathsf{T}}\mathbf{X}}. \tag{4}$$

If $\mathbf{Y}$ is $m \times n$ and $\mathbf{X}$ is $p \times q$, then this matrix is $mn \times pq$. These derivatives satisfy the chain rule, which we will use frequently, so that

$$\frac{d\mathbf{y}}{d\mathbf{z}^{\mathsf{T}}} = \frac{d\mathbf{y}}{d\mathbf{x}^{\mathsf{T}}}\frac{d\mathbf{x}}{d\mathbf{z}^{\mathsf{T}}}. \tag{5}$$

Delaying reproduction by redistributing fertility

The goal of this paper is to explore the population-level effects of delayed reproduction. We start with the population projection matrix

$$\mathbf{A} = \mathbf{U} + \mathbf{F}, \tag{6}$$

where $\mathbf{U}$ contains survival probabilities on the subdiagonal and zeros elsewhere, and $\mathbf{F}$ contains age-specific fertilities on the first row and zeros elsewhere. Let $\mathbf{f}$ be the vector of age-specific fertilities, in units of female children per female. Delayed reproduction will be modelled by a change in $\mathbf{f}$.

Let ξ denote some demographic outcome of interest (e.g., the population growth rate) that can be calculated from $\mathbf{A}$ and that depends on the fertility schedule. Using the chain rule of equation (A.5), the sensitivity of ξ to changes in the fertility schedule is

$$\frac{d\xi}{d\mathbf{f}^{\mathsf{T}}} = \left(\frac{d\xi}{d\text{vec}^{\mathsf{T}}\mathbf{A}}\right)\left(\frac{d\text{vec}\mathbf{A}}{d\text{vec}^{\mathsf{T}}\mathbf{F}}\right)\left(\frac{d\text{vec}\mathbf{F}}{d\mathbf{f}^{\mathsf{T}}}\right). \tag{7}$$

The marginal effect of a change $\Delta\mathbf{f}$ in reproduction on ξ is, to first order,

$$\Delta\xi = \frac{d\xi}{d\mathbf{f}^{\mathsf{T}}}\Delta\mathbf{f}. \tag{8}$$

We need to create a change $\Delta\mathbf{f}$ in fertility that represents a specified degree of delay. Because we want the effect of delay alone, all else being equal, we need a formulation that changes only the timing of reproduction. Simultaneously delaying and reducing fertility would reveal nothing about the effects of delay per se. Hence, the manipulations here hold TFR fixed.

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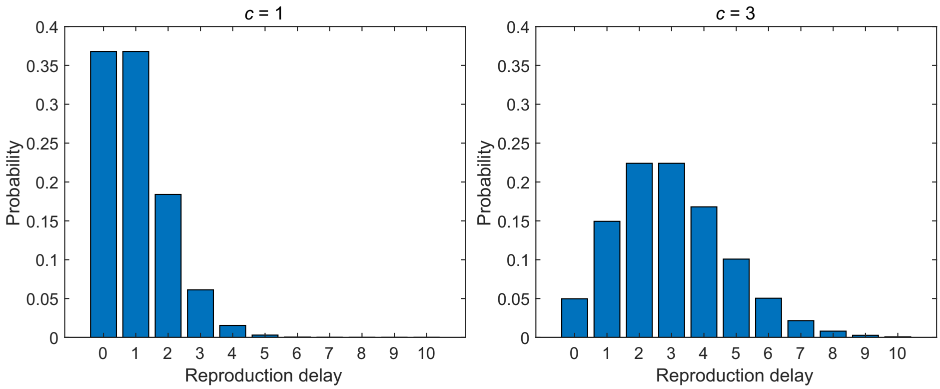
Figure 1 The distribution of reproductive delay produced by the diffusion operator **P** for $c = 1$ and $c = 3$



One way to delay reproduction is to apply a fixed delay to the fertility schedule, so that all fertility at age x is shifted to a specified later age (e.g., age $x + 1$). This approach was used by Dublin and Lotka (1925) and Caswell and Hastings (1980). Here we consider a more flexible description of delayed reproduction by creating a distributed delay. In this calculation, some of the fertility at age x remains at age x, while some is delayed to $x + 1$, some is delayed to $x + 2$, and so on. We do this using a diffusion process. We define a continuous-time biased random walk with transition rate matrix **Q** given by (supposing that $\omega = 4$)

$$\mathbf{Q} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{9}$$

The component of fertility at age x is delayed to age $x + 1$ at the rate 1. The random walk is absorbing at the upper end, since fertility cannot be delayed beyond age ω.

The discrete delay matrix is the matrix exponential[4] of $c\mathbf{Q}$

$$\mathbf{P} = e^{c\mathbf{Q}}. \tag{10}$$

The value of $c \geq 0$ determines the mean reproductive delay. The distributions of reproductive delay for $c = 1$ and $c = 3$ are shown in Figure 1. Delayed reproduction is defined by the transformation

$$\mathbf{f} \mapsto \mathbf{P}\mathbf{f} \tag{11}$$

and then

$$\Delta\mathbf{f} = \mathbf{P}\mathbf{f} - \mathbf{f} \tag{12}$$

which maintains the TFR unchanged, as desired.

4 The matrix exponential of a matrix X is defined as $e^X = \sum_{n=0}^{\infty} \frac{X^n}{n}$. However, this expression should not be used to compute the function (Moler and Van Loan, 2003). Matlab and R both implement reliable algorithms for the calculation.

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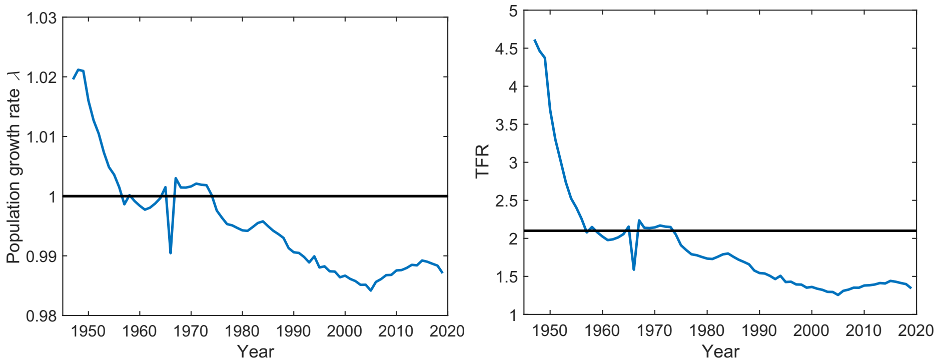
Figure 2 The period population growth rate (λ) and the total fertility rate (TFR) of Japan from 1947 to 2019. The horizontal lines mark the thresholds for population decline ($\lambda = 1$ and TFR = 2.1)



Delayed reproduction during a demographic transition in Japan

As an example of the trend towards increased survival and declining fertility in high income countries, I will analyse the case of Japan from 1947 to 2019 (rates from HMD 2022 and HFD 2022). Over this time span, period life expectancy increased from 54 to 87 years, the TFR declined from 4.6 to 1.3 children per woman, and the population growth rate declined from $\lambda = 1.020$ to 0.987. Both indices declined to below the replacement level, as shown in Figure 2. This sequence provides a convenient example, but similar results would follow from any similar time series. Comparative studies will be interesting.

Effects on the population growth rate

The population growth rate λ is given by the dominant eigenvalue of $\mathbf{A}$ and the right and left eigenvectors $\mathbf{w}$ and $\mathbf{v}$ give the stable age distribution and the reproductive value distribution, respectively. The eigenvectors are scaled so that

$$\sum_i w_i = 1 \tag{13}$$

$$\mathbf{w}^{\mathsf{T}}\mathbf{v} = 1. \tag{14}$$

The fertility matrix can be written

$$\mathbf{F} = \mathbf{e}_1 \mathbf{f}^{\mathsf{T}} \tag{15}$$

and thus

$$\frac{d\text{vec}\mathbf{F}}{d\mathbf{f}^{\mathsf{T}}} = (\mathbf{I}_\omega \otimes \mathbf{e}_1). \tag{16}$$

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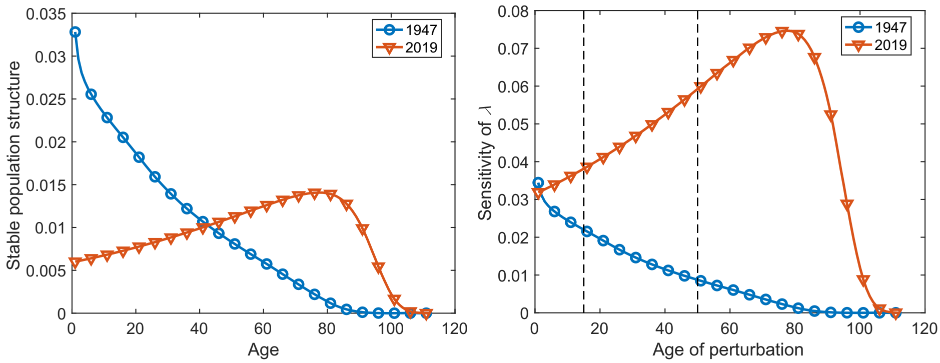
Figure 3 Left: Stable age distribution under rates in 1947, 1977 and 2019. Right: The sensitivity $d\lambda/d\mathbf{f}^{\mathrm{t}}r$ of the population growth rate to changes in age-specific fertility in 1947 and 2019. Vertical dashed lines demarcate the ages at which fertility is positive



Using the chain rule, the sensitivity of λ to changes in $\mathbf{f}$ is

$$\frac{d\lambda}{d\mathbf{f}^{\mathrm{T}}} = \left(\frac{d\lambda}{d\mathrm{vec}^{\mathrm{T}}\mathbf{A}}\right)\left(\frac{d\mathrm{vec}\mathbf{A}}{d\mathrm{vec}^{\mathrm{T}}\mathbf{F}}\right)\left(\frac{d\mathrm{vec}\mathbf{F}}{d\mathbf{f}^{\mathrm{T}}}\right). \tag{17}$$

Using the expression for the derivative of λ with respect to $\mathbf{A}$,

$$\frac{d\lambda}{d\mathrm{vec}^{\mathrm{T}}\mathbf{A}} = \left(\mathbf{w}^{\mathrm{T}} \otimes \mathbf{v}^{\mathrm{T}}\right) \tag{18}$$

(for derivation see Caswell, 2010) yields

$$\frac{d\lambda}{d\mathbf{f}^{\mathrm{T}}} = \left(\mathbf{w}^{\mathrm{T}} \otimes \mathbf{v}^{\mathrm{T}}\mathbf{e}_1\right) \tag{19}$$

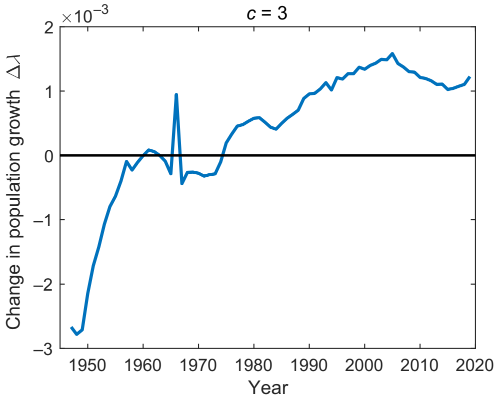
$$= v_1\mathbf{w}^{\mathrm{T}}. \tag{20}$$

That is, as is well known, the sensitivity of λ to a change in fertility (i.e., the marginal effect on population growth) is proportional to the stable age distribution. Figure 3 shows the stable age distribution and the sensitivity of λ to age-specific fertility for the increasing population of 1947 and the declining population of 2019. The difference between a strictly declining sensitivity when $\lambda > 1$ and a strongly increasing trend when $\lambda < 1$ is striking. It implies that, in a declining population, a unit change in fertility at later ages has greater effect than the same change at younger ages.

Next, we shift fertility later by an average of c years, as given in equation (11), so that $\Delta f = \mathbf{Pf} - \mathbf{f}$. The effect of this shift on population growth is

$$\Delta\lambda = \frac{d\lambda}{d\mathbf{f}^{\mathrm{T}}}\Delta\mathbf{f} \tag{21}$$

Figure 4 The change in the population growth rate, $\Delta\lambda$, of Japan due to delayed reproduction with $c = 3$



and is shown in Figure 4. The pattern is the inverse of that for the population growth rate shown in Figure 2. When $\lambda < 1$, the effect of delayed reproduction is positive rather than negative as is often assumed. As the population growth rate has continued to decline since the 1970s, the magnitude of the effect has increased.

Effects on population structure

Declining fertility is a concern because it leads to population aging. Does delayed reproduction have the same effect? We explore this question by calculating the effect of delayed fertility on the stable age distribution **w**. The sensitivity of **w** to changes in age-specific fertility follows from the general results given in Caswell (2019, Section 10.5.1, especially equation (10.73)), as

$$\frac{d\mathbf{w}}{d\mathbf{f}^{\mathsf{T}}} = \left(\lambda \mathbf{I} - \mathbf{A} + \mathbf{w}\mathbf{1}^{\mathsf{T}}\mathbf{A}\right)^{-1}\left(\mathbf{w}^{\mathsf{T}} \otimes \mathbf{I} - \mathbf{w}^{\mathsf{T}} \otimes \mathbf{w}\mathbf{1}^{\mathsf{T}}\right)\frac{d\mathrm{vec}\mathbf{A}}{d\mathbf{f}^{\mathsf{T}}}. \tag{22}$$

Figure 5 shows the change in the population structure resulting from delayed reproduction with $c = 3$, under 1947 rates and 2019 rates. In 1947, when population growth was positive, delayed fertility would have reduced the proportion of younger ages and increased the proportion of older ages. In other words, delayed reproduction would have led to population ageing. In 2019, when population growth was negative, delayed reproduction would have increased the proportion of young ages and reduced the proportion of older ages, which would have led to the population becoming younger.

Another way to look at population aging is to plot the effect of delayed reproduction on the fraction of the population over age 65. Figure 5 shows this effect over time, from 1947 to 2019. In the years when population growth was negative, delayed fertility would have reduced this fraction of the population. In the years when population growth was positive, the effect would have been reversed.

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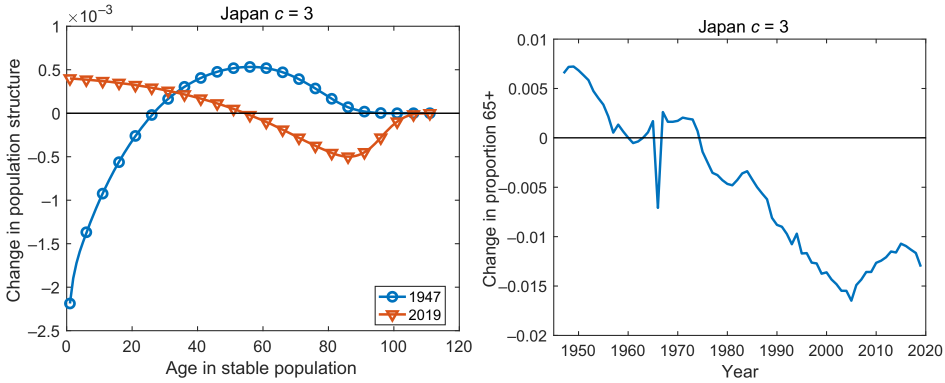
Figure 5 Left: Change in the age composition of the stable age distribution due to delayed reproduction under 1947 rates and 2019 rates. Right: Change due to delayed reproduction in the proportion aged 65+ in the stable population



Effects on immigration-subsidised equilibria

A population with fertility below the replacement level (more accurately, with combined fertility and mortality schedules that leave it with a population growth rate of $\lambda < 1$) can be maintained at an equilibrium by a constant immigration input. The model for the dynamics of such a population is

$$\mathbf{n}(t+1) = \mathbf{A}\mathbf{n}(t) + \mathbf{b} \tag{23}$$

where $\mathbf{b}$ is a vector giving the age structure of immigrants.[5] A non-negative equilibrium population $\hat{\mathbf{n}}$, called a stationary-through-immigration population by Schmertmann (2012), exists provided that $\lambda < 1$. That equilibrium satisfies

$$\hat{\mathbf{n}} = \mathbf{A}\hat{\mathbf{n}} + \mathbf{b} \tag{24}$$

so that

$$\hat{\mathbf{n}} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}. \tag{25}$$

The equilibrium depends on the age structure of the immigration vector $\mathbf{b}$. To provide an example, I set $\mathbf{b}$ proportional to the 2000–2025 World Health Organization standard population (Ahmad et al., 2001).[6] However, the analysis could be carried out with any age structure of immigration. Figure 6 shows the age structures of the equilibrium population under the rates in 1976, 1986, and 2016 (remember that the equilibrium only exists when $\lambda < 1$). Because the model is linear, the equilibrium abundance can be scaled at will; here I scale immigration so that $\|\mathbf{b}\| = 1$.

5 Such populations have been called "subsidised" because they are maintained by a subsidy of migrants (Caswell, 2019; Pascual and Caswell, 1991, Section 10.4).
6 Obtained from https://seer.cancer.gov/stdpopulations/world.who.html

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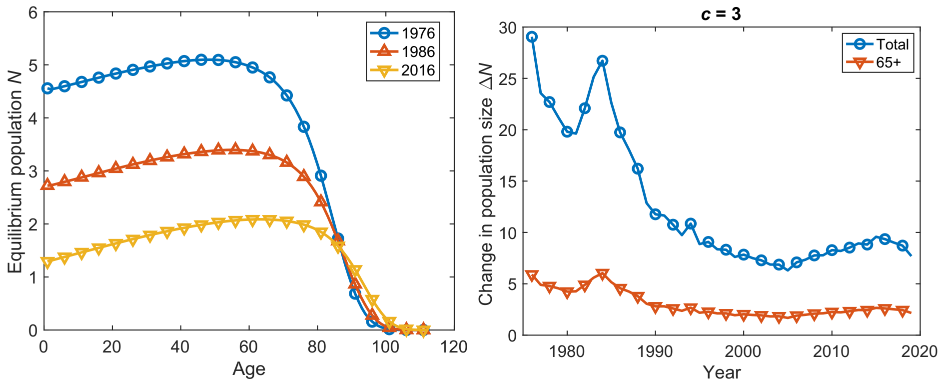
Figure 6 Left: Equilibrium population age structure $\hat{n}$ under the rates in 1976, 1986 and 2016. Right: Change in the equilibrium population size and the equilibrium population size at ages 65+, due to delayed reproduction



The sensitivity of $\hat{\mathbf{n}}$ follows from Caswell (2019, Equation 10.58),

$$\frac{d\hat{\mathbf{n}}}{d\mathbf{f}^{\mathsf{T}}} = (\mathbf{I} - \mathbf{A})^{-1}\left(\hat{\mathbf{n}}^{\mathsf{T}} \otimes \mathbf{e}_1\right). \tag{26}$$

We can consider the effect of delayed reproduction on total equilibrium population size $\hat{N} = 1^{\mathsf{T}}\hat{\mathbf{n}}$, which is

$$\Delta\hat{N} = 1^{\mathsf{T}} \frac{d\hat{\mathbf{n}}}{d\mathbf{f}^{\mathsf{T}}} \Delta f. \tag{27}$$

Figure 6 shows the impact of a delay with a mean of three years ($c = 3$) over the 1976–2019 period, in which the population growth rate was consistently negative. During this period, the effect of delayed fertility on the population size was consistently positive.

Similarly, we can calculate the effect of delayed reproduction on the age structure of the equilibrium population. For example, consider the proportion D of the population over some age (65, for instance). Then

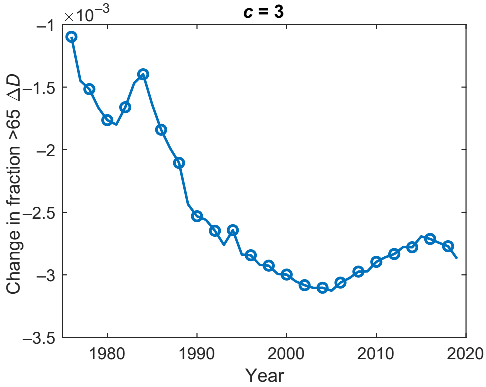
$$D = \frac{\mathbf{a}^{\mathsf{T}}\hat{\mathbf{n}}}{1^{\mathsf{T}}\hat{\mathbf{n}}} \tag{28}$$

where $\mathbf{a}$ is a vector with ones in entries from 65 to ω and zeros elsewhere. The sensitivity of D to changes in $\mathbf{f}$ is

$$\frac{dD}{d\mathbf{f}^{\mathsf{T}}} = \left(\frac{1^{\mathsf{T}}\hat{\mathbf{n}}\mathbf{a}^{\mathsf{T}} - \mathbf{a}^{\mathsf{T}}\hat{\mathbf{n}}1^{\mathsf{T}}}{(1^{\mathsf{T}}\hat{\mathbf{n}})^2}\right) \tag{29}$$

(Caswell, 2019, Equation 10.33). The change ΔD in this over-65 fraction due to delayed reproduction is shown in Figure 7. The effects were consistently negative over the time period when the population growth rate was been below the replacement level.

Figure 7 Change due to delayed reproduction in the fraction of the equilibrium population aged 65+



In sum, delayed reproduction in a stationary-through-immigration population increases the equilibrium population size and reduces the proportional representation of old individuals, making the population younger.

Effects of delayed parity progression

Delayed reproduction may, of course, be more complicated than just a shift of age-specific fertility. Fertility is also affected by parity, and age-specific parity transition matrices capture those effects. Caswell (2020) developed a multistate age × parity model as part of a matrix kinship model. The details of the multistate models are laid out in Caswell et al. (2018). Very briefly, individuals are classified into ω age classes and s parity stages (in the HFD, $s = 6$). The state of the population is given by the block-structured vector

$$\tilde{\mathbf{n}} = \begin{pmatrix} n_{11} \\ \vdots \\ n_{s1} \\ \hline \vdots \\ \hline n_{1\omega} \\ \vdots \\ n_{s\omega} \end{pmatrix}. \tag{30}$$

The vector is projected by a projection matrix constructed from four sets of matrices, which specify transitions among parity classes, survival from one age class to the next, fertility as the result of a transition from one parity class to the next and the assignment of new births to age class 1 and parity 0.

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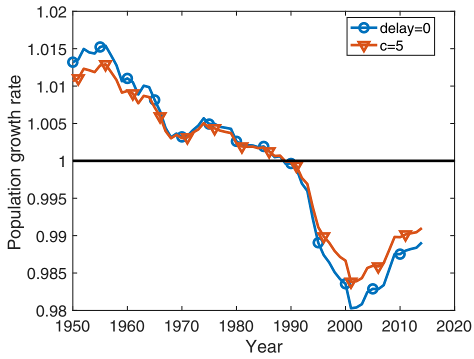
Figure 8 The population growth rate λ for Slovakia, calculated from the age-parity model of Caswell (2020), with and without delayed reproduction



Using the protocol given by Caswell (2012) and Caswell et al. (2018) yields a block-structured population projection matrix

$$\tilde{\mathbf{A}} = \tilde{\mathbf{U}} + \tilde{\mathbf{F}} \tag{31}$$

with blocks arranged in the same form as the familiar age-classified Leslie matrix, which is a familiar structure in multiregional demography (e.g., Feeney, 1970; Rogers, 1995). The population growth rate λ and stable structure $\tilde{\mathbf{w}}$ (age $\times$ parity) are calculated from $\tilde{\mathbf{A}}$ in the usual way.

Caswell (2020) developed an age $\times$ parity model using demographic data from the HMD and HFD for Slovakia from 1950 to 2014. The case of Slovakia was chosen because it had one of the longest sequence of parity information and steepest declines in fertility in the database. As shown in Figure 8, the population growth rate in Slovakia dropped below the replacement level in 1989 and remained below the replacement level thereafter.

Of the several ways that delayed reproduction could be implemented in this model, we consider here a distributed delay, with a specified value of c, applied to the entire age-specific parity transition structure. The following figures, which are more suggestive than definitive, show some results.

Figure 8 shows the impact of a delay in reproduction on the population growth rate. Because the effects for Slovakia are smaller in magnitude, for clarity the results are shown for $c = 5$. When population growth rate dropped below the replacement level, the effect of delayed reproduction changed from negative to positive, just as it did in the analyses based on age-specific, as opposed to age × parity-specific, fertility.

Figure 9 shows the effect of delayed reproduction on the stable age distribution (the marginal age distribution from the stable age × parity distribution), expressed as the proportion of the population over age 65. When population growth was above the replacement level, delayed reproduction led to an older population. Conversely, when population growth was below the replacement level, delayed reproduction had the opposite effect.

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Figure 9 The fraction of the stable marginal age distribution over the age of 65 for Slovakia, calculated from the from age × parity model of Caswell (2020), with and without delayed reproduction

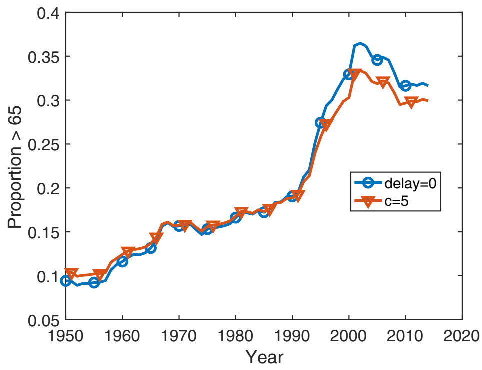
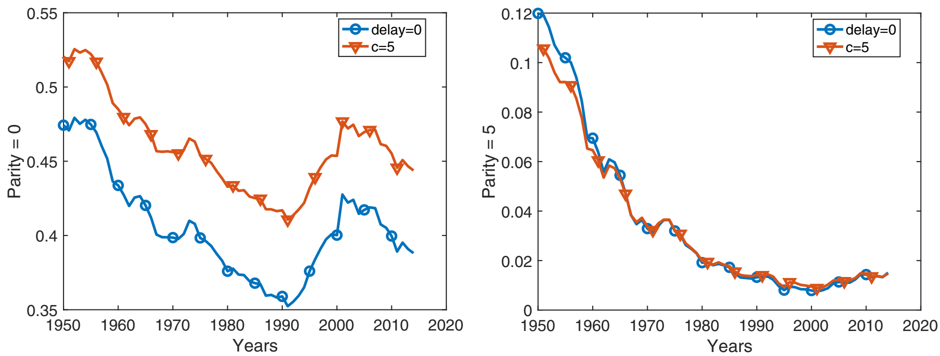


Figure 10 The fraction of the population in the parity 0 and parity 5 classes in the stable marginal parity distribution, with and without delayed reproduction



Finally, Figure 10 shows the components of the marginal parity distribution for parity 0 and parity 5 (the highest parity included in the HFD data). Delayed reproduction increased the proportion of parity 0, and had little effect on the proportion of parity 5.

Discussion

"Everyone knows" that delayed reproduction can be expected to reduce population growth and population size, and make age distributions older. But this is not always the case. The effects of delayed reproduction differ qualitatively depending on whether the population is increasing or decreasing. Early authors sometimes recognised this, but their attention was focused on situations of positive, and even rapid, growth. Today, when much of the world

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has demographic rates below the replacement level, this received wisdom must be challenged. The methods presented here have potential applications to other populations, other models, and other response variables.

Extensions

A number of extensions present themselves immediately. First, population growth rate, stable population structure, and equilibrium population size are all asymptotic long-term properties. Delayed reproduction can also affect short-term transient dynamics, and these effects are likely to be of considerable interest. Sensitivity analysis results exist for transient dynamics (Caswell, 2007, 2019, Chapter 7). The usual population projections prepared by statistical offices are projections of transient dynamics under projected future mortality, fertility, and immigration rates. The sensitivity analysis of such projections (Caswell and Sanchez Gassen, 2015) can extend the results here to such transient dynamics.

Second, the comparison of results using age-specific fertility and age × parity-specific fertility warrants further investigation. In addition to shifting the entire parity progression matrix, as was done here, reproduction could be delayed by changing the rates of various parity transitions. This will allow for the investigation the “longer” part of the “later, longer, fewer” slogan, since age × parity models explicitly include the rates of progression from one parity class to the next.

Third, the model presented here is deterministic. In a series of papers, Tuljapurkar and colleagues analysed the effects of delayed reproduction in a stochastic environment (Tuljapurkar, 1990; Tuljapurkar and Istock, 1993; Tuljapurkar and Wiener, 2000). They showed that in a deterministic environment, delayed reproduction reduces the population growth rate (they noted that this was true in increasing populations, but did not pursue the case of declining populations). In their examination of a variety of life cycles, some including processes (dormancy, diapause) that do not occur in humans, they found that in stochastic environments, delayed reproduction could sometimes increase, rather than decrease, the stochastic population growth rate. Extending these results to equilibrium or declining populations is an open research problem.

Theory and calculation

The questions addressed here are theoretical; they consider the effects of a delay in reproduction, all else being equal. That *ceteris paribus* clause includes preserving the lifetime quantity of reproduction. Of course this is not the only possible way in which people of different ages could make decisions about the timing of reproduction. The fertility shift, modelled here as a directional diffusion, could be extended to other kinds of perturbations. Some earlier literature focused on the relative impact of timing and amount of reproduction (Caswell and Hastings, 1980; Lewontin, 1965) and it would be interesting to extend these analyses in this direction. The effects of delayed reproduction seem small, but their

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size can only be judged in comparison to the effects of other perturbations. This is also an open research problem.

The title of this paper describes the effects of delayed reproduction in declining populations as “surprising”. Of course, a result that is surprising to one person may be nothing new to another. But I believe that the qualitative changes reported here have not been recognised in discussions of the effects of delayed reproduction. The methods presented here have the potential to open up additional demographic outcomes to analysis and can be used to explore the effects of fertility changes in populations with below-replacement fertility levels.

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Appendix

Matrix calculus

Matrix calculus permits the consistent differentiation of scalar-, vector-, and matrix-valued functions of scalar, vector, or matrix arguments. For the convenience of the reader, this appendix[7] presents a brief statement of the essential results. More detail, and many demographic applications, can be found in Caswell (2019).

If x and y are scalars, the derivative of $\mathbf{y}$ with respect to x is the familiar derivative dy/dx. If $\mathbf{y}$ is a $n \times 1$ vector and x is a scalar, the derivative of y with respect to x is the $n \times 1$ vector

$$\frac{d\mathbf{y}}{dx} = \begin{pmatrix} \frac{dy_1}{dx} \\ \vdots \\ \frac{dy_n}{dx} \end{pmatrix}. \tag{A.1}$$

If y is a scalar and $\mathbf{x}$ is a $m \times 1$ vector, the derivative of y with respect to x is the $1 \times m$ gradient vector

$$\frac{dy}{d\mathbf{x}^{\mathsf{T}}} = \begin{pmatrix} \frac{\partial y}{\partial x_1} & \cdots & \frac{\partial y}{\partial x_m} \end{pmatrix}. \tag{A.2}$$

Note the orientation of $d\mathbf{y}/dx$ as a column vector and $dy/d\mathbf{x}^{\mathsf{T}}$ as a row vector.

If $\mathbf{y}$ is a $n \times 1$ vector and $\mathbf{x}$ a $m \times 1$ vector, the derivative of $\mathbf{y}$ with respect to $\mathbf{x}$ is the $n \times m$ Jacobian matrix

$$\frac{d\mathbf{y}}{d\mathbf{x}^{\mathsf{T}}} = \left(\frac{dy_i}{dx_j}\right). \tag{A.3}$$

Derivatives involving matrices are written by transforming the matrices into vectors using the vec operator (which stacks the columns of the matrix into a column vector), and then applying the rules for vector differentiation. Thus, the derivative of the $m \times n$ matrix $\mathbf{Y}$ with respect to the $p \times q$ matrix $\mathbf{X}$ is the $mn \times pq$ matrix

7 This appendix is modified from Section 2 of Caswell (2008) under the terms of a Creative Commons Attribution license.

$$\frac{d\text{vec}\mathbf{Y}}{d\text{vec}^{\mathsf{T}}\mathbf{X}}. \tag{A.4}$$

For notational convenience, I will write $\text{vec}^{\mathsf{T}}\mathbf{X}$ for $(\text{vec}\mathbf{X})^{\mathsf{T}}$.

These definitions (unlike some alternatives; see Magnus and Neudecker, 1985) lead to the familiar chain rule. If $\mathbf{Y}$ is a function of $\mathbf{X}$ and $\mathbf{X}$ is a function of $\mathbf{Z}$, then

$$\frac{d\text{vec}\mathbf{Y}}{d\text{vec}^{\mathsf{T}}\mathbf{Z}} = \frac{d\text{vec}\mathbf{Y}}{d\text{vec}^{\mathsf{T}}\mathbf{X}} \frac{d\text{vec}\mathbf{X}}{d\text{vec}^{\mathsf{T}}\mathbf{Z}}. \tag{A.5}$$

The derivatives of matrices are constructed by forming the differentials of the expressions involving the matrices. The differential of a matrix (or vector) is the matrix (or vector) containing the differentials of the elements, i.e.,

$$d\mathbf{X} = (dx_{ij}). \tag{A.6}$$

If, for vectors $\mathbf{x}$ and $\mathbf{y}$ and some matrix Q, it can be shown that

$$d\mathbf{y} = \mathbf{Q}d\mathbf{x} \tag{A.7}$$

then

$$\frac{d\mathbf{y}}{d\mathbf{x}^{\mathsf{T}}} = \mathbf{Q}. \tag{A.8}$$

This is the “first identification theorem” of Magnus and Neudecker (1985), see also Neudecker (1969) and it is fundamental to calculating derivatives.

The combination of the chain rule and the identification theorem permits more complicated expressions involving differentials to be turned into derivatives with respect to an arbitrary vector, say $\mathbf{u}$. If

$$d\mathbf{y} = \mathbf{Q}d\mathbf{x} + \mathbf{R}d\mathbf{z} \tag{A.9}$$

then

$$\frac{d\mathbf{y}}{d\mathbf{u}^{\mathsf{T}}} = Q\frac{d\mathbf{x}}{d\mathbf{u}^{\mathsf{T}}} + \mathbf{R}\frac{d\mathbf{z}}{d\mathbf{u}^{\mathsf{T}}} \tag{A.10}$$

for any $\mathbf{u}$.

We will make extensive use the Kronecker product, defined as

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{A.11}$$

The vec operator and the Kronecker product are related, if

$$\mathbf{Y} = \mathbf{ABC} \tag{A.12}$$

then

$$\text{vec}\mathbf{Y} = (\mathbf{C}^{\mathsf{T}} \otimes \mathbf{A})\text{vec}\mathbf{B}. \tag{A.13}$$

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