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Connecting in the Dirac Equation the Clifford Algebra of Lorentz Invariance with the Lie Algebra of $SU(N)$ Gauge Symmetry

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Abstract: In this paper, we study possible mathematical connections of the Clifford algebra with the $su(N)$ -Lie algebra, or in more physical terms the links between space-time symmetry (Lorentz invariance) and internal $SU(N)$ gauge-symmetry for a massive spin one-half fermion described by the Dirac equation. The related matrix algebra is worked out in particular for the $SU(2)$ symmetry and outlined as well for the color gauge group $SU(3)$. Possible perspectives of this approach to unification of symmetries are briefly discussed. The calculations make extensive use of tensor multiplication of the matrices involved, whereby our focus is on revisiting the Coleman–Mandula theorem. This permits us to construct unified symmetries between Lorentz invariance and gauge symmetry in a direct product sense.

Keywords: extended Dirac equation; isospin; Clifford algebra; $SU(N)$ symmetry



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1. Introduction

Modern non-abelian gauge field theory started with the seminal paper by Yang and Mills [1] in 1954, when they studied the conservation of isotopic spin and the associated $SU(2)$ gauge invariance. This subject was investigated in connection with the idea to assemble the proton and neutron in a doublet to describe the nuclear force. Their emphasis was on the gauge field equations, which resembled those of the electromagnetic field yet revealed new non-linear couplings due to the algebraic properties of the symmetry group involved. Quantum Yang–Mills theory has developed ever since into a cornerstone of the modern Standard Model (SM) of elementary particle physics [2–4].

Here, we will start from scratch concerning the fermion sector of the SM and determine various links of the general $SU(N)$ symmetry with the properties of the fermion Clifford algebra, as it is induced by the Lorentz transformation in Minkowski space-time. An explicit example is then given in a subsequent section illustrating the mathematical approach by which the symmetry group $SU(2)$ can be incorporated in the standard Dirac equation in the Weyl basis. The key point is that, instead of rotating the multiplet of Dirac spinors under $SU(N)$, one can equally well tensor-multiply the gamma matrices from the right side by the related unit matrix 1_N to accommodate the associated N -fold multiplet, thus expanding the Dirac spinor to a $4N$ -component spinor. Thus, the essential question is then: Is there a unitary transformation connecting these two versions of the Dirac equation? The answer is yes, which is the main accomplishment of this paper, and the procedure for how to obtain the required transformation will be described in detail. This approach can easily be extended to products of symmetry groups for which an example is also presented in the last section, and also permits us to construct unified symmetries between Lorentz invariance and gauge symmetry in a direct product sense in compliance with the Coleman–Mandula theorem [5].

2. The Standard Dirac Equation in the Weyl Basis

It is well known that the Lie algebra for the Lorentz group [2–4,6–9] can be decomposed into two commuting sub-algebras, such that $so(3,1) = su(2) \oplus su(2)$. Their elements define the generators of the $SU(2) \oplus SU(2)$ representation of the LG. These algebraic relations are constitutive for any other representation of the LG. The fundamental representation of the $SU(2)$ group in terms of the Pauli [10] matrix vector plays a key role in the representation of the Lorentz group. These matrices read as follows

$$\sigma = \left(\left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \right). \quad (1)$$

In this section, we will closely follow the reasoning and nomenclature of the previous work by Marsch and Narita [11–14]. The Pauli matrices obey $\sigma_x \sigma_y \sigma_z = i1_2$ and mutually anticommute with each other. In addition, $\sigma_j^2 = 1_2$ for $j = 1, 2, 3$. Using the algebra of these matrices, we can write the Dirac gamma matrices in the Weyl basis as follows

$$\gamma^\mu = (\gamma_0, \boldsymbol{\gamma}) = (\sigma_x \otimes 1_2, i\sigma_y \otimes \boldsymbol{\sigma}), \quad (2)$$

where the symbol \otimes stands for the tensor product of the involved matrices. The properties of the above Pauli matrices guarantee the validity of the Clifford algebra

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} 1_4, \quad (3)$$

which ensures Lorentz invariance. Conventionally, the physical spin-doublet degree of freedom is described by the three-vector of the sigma matrices on the right, whereas the particle-antiparticle degree of freedom is taken care of by the Pauli matrices on the left side of the above tensor product. For completeness we quote finally the Dirac equation

$$\gamma^\mu i\partial_\mu \psi = m\psi, \quad (4)$$

acting on the four-component Dirac spinor ψ for a spin-one-half fermion of mass m . We use units in which the speed of light $c = 1$, and the Planck constant is unity as well, which are convenient in quantum field theory to ease the notation. Here, as usually, $\partial_\mu = (\partial_t, \partial_x)$ is the covariant derivative in the Minkowski space-time of special relativity. How can any other internal additional degrees of freedom be considered adequately, for example the N degrees of freedom as described by the general $SU(N)$ symmetry group? This issue is dealt with in the next section.

3. Connecting $SU(N)$ Symmetry with the Dirac Equation

According to modern Yangs–Mills theory [1,4] for a multiplet of N fermions, these particles are assumed to transform under the $SU(N)$ symmetry group as a complex N -vector Ψ with N Dirac spinors as entries. Each generator G^a of the group is associated with a bosonic vector field $A_\mu^a(x)$, with $x = x^\mu = (t, \mathbf{x})$, and these fields enter the Dirac equation as a connection in the covariant derivative in the form

$$D_\mu = \partial_\mu - i g A_\mu^a(x) G^a, \quad (5)$$

whereby g is the single coupling constant, and the superscript a runs over the integers numbering the generators of the group. As usual convention, a is to be summed over when appearing twice. For given N there are $N^2 - 1$ generators of a particular $SU(N)$ Lie group, so that a runs from 1 to $N^2 - 1$. Any general group element can then be written as exponential

$$E(x) = \exp(i\theta^a(x) G^a), \quad (6)$$

where $\theta^a(x)$ are parameter functions of x in the case of local $SU(N)$ symmetry, or constants in the case of global symmetry. The complex N -vector reads

$$\Psi^\dagger = (\psi_1^\dagger, \psi_2^\dagger, \dots, \psi_N^\dagger), \tag{7}$$

and the related gamma matrix is to be defined as

$$\Gamma^\mu = \mathbf{1}_N \otimes \gamma^\mu, \tag{8}$$

so that the standard Dirac gamma matrices are acting on each component of the spinor multiplet, and therefore we have $\Gamma^\mu i\partial_\mu \Psi = m\Psi$. Each fermion in the multiplet is assumed here to have the same mass m , which may of course also be set to zero, which is normally done in the Standard Model (SM). The coupling of the fermions to the gauge fields is given by the conventional Dirac equation involving the covariant derivate (5) as follows

$$\Gamma^\mu iD_\mu \Psi = m\Psi. \tag{9}$$

Our key goal in this paper was to find a way to transform (9) into another new form in which the gamma matrices can be written

$$\tilde{\Gamma}^\mu = \gamma^\mu \otimes \mathbf{1}_N, \tag{10}$$

and then the Dirac equation attains the formally similar appearance as above

$$\tilde{\Gamma}^\mu iD_\mu \tilde{\Psi} = m\tilde{\Psi}. \tag{11}$$

Yet, the physical interpretation is different, because Ψ is an N -vector with standard Dirac spinors as elements, whereas $\tilde{\Psi}$ is a single Dirac spinor, yet each of its four components is an N -vector (of complex-numbers) reflecting the degrees of freedom associated with the dimension N of the $SU(N)$ symmetry group. The key question is then: Is there a unitary transformation connecting these two versions of the Dirac equation? Before we discuss this in more detail, we consider the simplest example of a non-abelian symmetry.

4. The $SU(2)$ Symmetry as Explicit Example

Therefore, let us consider, due to its transparency, the $SU(2)$ symmetry explicitly. Then, according to (8), we can write the Dirac gammas in extended form in terms of 8×8 matrices as follows

$$\Gamma^\mu = \left(\left(\begin{matrix} 0 & 1_2 & 0 & 0 \\ 1_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_2 \\ 0 & 0 & 1_2 & 0 \end{matrix} \right), \left(\begin{matrix} 0 & \sigma & 0 & 0 \\ -\sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{matrix} \right) \right), \tag{12}$$

By means of the unitary transformation V (involving U that exchanges the inner columns and rows) given by the 8×8 matrix

$$V = U \otimes 1_2 = \left(\begin{matrix} 1_2 & 0 & 0 & 0 \\ 0 & 0 & 1_2 & 0 \\ 0 & 1_2 & 0 & 0 \\ 0 & 0 & 0 & 1_2 \end{matrix} \right), \tag{13}$$

and which the U obeys $V^\dagger = V^{-1} = V$, we can cast the gamma matrices in a form similar to that we introduced already in (2). One obtains

$$\hat{\Gamma}^\mu = (\sigma_x \otimes 1_2 \otimes 1_2, i\sigma_y \otimes 1_2 \otimes \sigma). \tag{14}$$

Here, $\hat{\Gamma}^\mu = V\Gamma^\mu V$, and thus $\hat{\Psi} = V\Psi$. Apparently, the two $SU(2)$ degrees of freedom appear now at a different location of the tensor products involved as compared with (12). However, the Equation (14) does not quite yet have the desired form as suggested in (10). To obtain it, we need to further unitarily transform $\hat{\Gamma}^\mu$ by help of $W = 1_2 \otimes U$ (with $W^{-1} = W^\dagger = W$). Then, we finally obtain

$$\tilde{\Gamma}^\mu = W\hat{\Gamma}^\mu W = (\sigma_x \otimes 1_2 \otimes 1_2, i\sigma_y \otimes \sigma \otimes 1_2) = \gamma^\mu \otimes 1_2. \tag{15}$$

The spin operator of the Dirac equation based on these gamma matrices is now given by $\mathbf{S} = \frac{1}{2}1_2 \otimes \sigma \otimes 1_2$, and there is a related isospin $\mathbf{I} = \frac{1}{2}1_4 \otimes \sigma$, which obviously commutes with the spin \mathbf{S} and the gamma matrices as defined in (15) above and directly corresponds to the generators of $SU(2)$. This result reproduces some of the derivations made by Marsch and Narita [14], who started from the extended Dirac equation established by them on the basis of the four-vector representation of the Lorentz group, but it goes somewhat further.

The general group element of $SU(2)$ can, by help of the isospin after (6), thus be written as the exponential

$$E(x) = \exp(i\lambda(x) \cdot \mathbf{I}), \tag{16}$$

which is nothing but the general phase permitted under the $SU(2)$ symmetry for the bi-spinor $\hat{\Psi}$. The angular parameter theta is now a three-vector. Written out explicitly, the spin operator reads

$$\mathbf{S} = \frac{1}{2} \left(\left(\begin{matrix} 0 & 1_2 & 0 & 0 \\ 1_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_2 \\ 0 & 0 & 1_2 & 0 \end{matrix} \right), \left(\begin{matrix} 0 & -i1_2 & 0 & 0 \\ i1_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i1_2 \\ 0 & 0 & i1_2 & 0 \end{matrix} \right), \left(\begin{matrix} 1_2 & 0 & 0 & 0 \\ 0 & -1_2 & 0 & 0 \\ 0 & 0 & 1_2 & 0 \\ 0 & 0 & 0 & -1_2 \end{matrix} \right) \right). \tag{17}$$

For comparison with the isospin \mathbf{I} , we can write the gamma matrices in the format reading

$$\tilde{\Gamma}^\mu = (\tilde{\Gamma}_0, \tilde{\Gamma}) = \left(\left(\begin{matrix} 0 & 1_4 \\ 1_4 & 0 \end{matrix} \right), \left(\begin{matrix} 0 & \sigma \otimes 1_2 \\ -\sigma \otimes 1_2 & 0 \end{matrix} \right) \right). \tag{18}$$

When multiplying the respective matrices mutually which each other, it becomes obvious that the commutator holds: $[\mathbf{I}, \tilde{\Gamma}^\mu] = 0$. Therefore, the isospin commutes with the Dirac kinetic operator $\tilde{\Gamma}^\mu i\partial_\mu$, and for constant three-vector θ the group element E as given in (16) commutes with it as well.

5. The Extended Dirac Equation Involving $SU(N)$

The example given in the previous section illustrates the mathematical approach by which the symmetry group $SU(2)$ can be formally incorporated in the standard Dirac equation. Yet, instead of rotating the doublet of Dirac spinors under $SU(2)$, we can also tensor-multiply the gamma matrices from the right side to include the doublet. Consequently, in the general case of $SU(N)$ we can, instead of using the common procedure described by (8), also expand the Pauli matrices by tensor multiplication of (1) with 1_N from the right, thus obtaining their trivially generalized form

$$\tilde{\sigma}_N = \left(\left(\begin{matrix} 0 & 1_N \\ 1_N & 0 \end{matrix} \right), \left(\begin{matrix} 0 & -i1_N \\ i1_N & 0 \end{matrix} \right), \left(\begin{matrix} 1_N & 0 \\ 0 & -1_N \end{matrix} \right) \right) = \sigma \otimes 1_N. \tag{19}$$

As a result, the Pauli spinor now is $\phi_N^\dagger = (\phi_1^\dagger(N), \phi_2^\dagger(N))$, whereby its two components become complex vectors transforming under $SU(N)$, i.e., $\phi^\dagger(N) = (c_1, c_2, \dots, c_N)$. Moreover, we get the same number of degrees of freedom as before, but formally the ones associated with $SU(N)$ are subsumed in the components of the spin vector $\tilde{\sigma}_N$. Therefore, the extended Dirac matrices now read

$$\tilde{\Gamma}^\mu = \gamma^\mu \otimes 1_N = (\sigma_x \otimes 1_2 \otimes 1_N, i\sigma_y \otimes \sigma \otimes 1_N) = (\sigma_x \otimes 1_{2N}, i\sigma_y \otimes \tilde{\sigma}_N). \tag{20}$$

The corresponding Clifford algebra takes the form

$$\tilde{\Gamma}^\mu \tilde{\Gamma}^\nu + \tilde{\Gamma}^\nu \tilde{\Gamma}^\mu = 2g^{\mu\nu} 1_{4N}, \quad (21)$$

and the spin (rotation) and rapidity (boost) operators attain the shape

$$\mathbf{S}_N = \frac{1}{2}(1_2 \otimes \sigma_N), \quad \mathbf{R}_N = \frac{i}{2}(-\sigma_z \otimes \sigma_N). \quad (22)$$

They obey the standard Lorentz algebra, yielding $\mathbf{S}_N \times \mathbf{S}_N = i\mathbf{S}_N$, $\mathbf{R}_N \times \mathbf{R}_N = -i\mathbf{R}_N$; $\mathbf{R}_N \times \mathbf{S}_N = \mathbf{S}_N \times \mathbf{R}_N = i\mathbf{R}_N$. Considering any $SU(N)$ generator G^a being an $N \times N$ matrix, we can define the related symmetry operator as a tensor-product matrix

$$S^a = 1_4 \otimes G^a, \quad (23)$$

which by definition commutes with all the relevant operators of the Dirac equation as given in the Equations (20) and (22). Thus, we have the important relations

$$[S^a, \tilde{\Gamma}^\mu] = 0, [S^a, \tilde{\Gamma}_5] = 0, [S^a, \mathbf{S}] = [S^a, \mathbf{R}] = 0. \quad (24)$$

They express the key requirement imposed on any quantum field theory by the famous Coleman–Mandula theorem [5], stating that “space-time and internal symmetries cannot be combined in any but a trivial way”, which is to say for our case that they must be connected as a direct product by simple tensor multiplication. Then, the general permitted phase \tilde{P} of the spinor $\tilde{\Psi}$ solving the expanded Dirac equation can be written as

$$\tilde{P}(x) = \exp(i\theta^a(x) S^a), \quad (25)$$

where the phase angle N -vector theta has N components which may depend on the space-time variable x for local $SU(N)$ symmetry, or be constant for global symmetry. As before, the index a runs from 1 to N . In analogy to (6), $\tilde{P}(x)$ is just the representation of any element of the $SU(N)$ Lie group. The generators S^a form a Lie algebra defined through the commutation relations

$$[S^a, S^b] = if^{abc} S^c, \quad (26)$$

with the known structure constants of the algebra [1,4], which are not affected by the trivial extension Formula (23). For $N = 2$, we just retain the angular momentum algebra of the isospin $\mathbf{I} = \frac{1}{2}1_4 \otimes \sigma$. The unitary transformation (13) given above was essential for obtaining the results of this section dealing with $SU(2)$ symmetry. We cannot explicitly give the general transformation for $SU(N)$, but for $N = 3$ related with the color symmetry of Quantum Chromodynamics (QCD) we also found the relevant unitary matrix.

6. The Unitary Transformation for $SU(3)$ Symmetry

The symmetry group $SU(3)$ plays a key role in the strong interactions of the SM [2–4]. Therefore, we discuss in this section the matrix that provides the required transformation from (8) to (10) for the Gamma matrices. Corresponding to (19), we first define the spin matrix vector

$$\sigma_N = 1_N \otimes \sigma. \quad (27)$$

Now we are dealing with the special case $\sigma_3 = 1_3 \otimes \sigma$ and $\tilde{\sigma}_3 = \sigma \otimes 1_3$. The sought for unitary transformation matrix X (with $X^{-1} = X^\dagger$) is given by the 6×6 -matrix that delivers the connection

$$X\sigma_{3x}X^\dagger = \tilde{\sigma}_{3x}. \quad (28)$$

After some tedious trials, we find that it has to be

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad X^{-1} = X^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (29)$$

Thus, we obtain the relations $X\sigma_{3x}X^\dagger = \tilde{\sigma}_{3x}$, $X\sigma_{3y}X^\dagger = -\tilde{\sigma}_{3y}$ and finally $X\sigma_{3z}X^\dagger = -\tilde{\sigma}_{3z}$. This yields as it should $\sigma_{3x}\sigma_{3y}\sigma_{3z} = i1_6$. Note that the sign in front of the y-component and z-component of sigma is just a phase factor being without importance for the algebra of the sigmas.

Let us, as a brief interlude, discuss another example for the extended Pauli matrix vector for the group $SU(3)$. Other than the definition of σ_3 above, we may consider the, at a first sight apparently less trivial, version

$$\hat{\sigma}_3 = \left(\left(\begin{pmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{pmatrix}, \begin{pmatrix} \sigma_y & 0 & 0 \\ 0 & \sigma_z & 0 \\ 0 & 0 & \sigma_x \end{pmatrix}, \begin{pmatrix} \sigma_z & 0 & 0 \\ 0 & \sigma_x & 0 \\ 0 & 0 & \sigma_y \end{pmatrix} \right) \right). \quad (30)$$

It has the same algebraic properties as the Pauli matrices (1), but manifestly reflects the three degrees of freedom of $SU(3)$. In the top position of the matrix diagonal, we have already the three Pauli matrices. By changing the basis for the middle and bottom positions, one can however transform their three components to attain the standard form as used in the top. Therefore, after adequate unitary transformations we can recover the original simple definition $\sigma_3 = 1_3 \otimes \sigma$, we started with previously. The highly symmetric form in (30) obtained by cyclic permutation is only possible for $SU(3)$, though, since the three dimensions of its matrices in fundamental representation correspond to the three axes spanning the real three-dimensional space, for the rotation of which the three Pauli matrices provide the generators.

Let us return to the original Dirac equation for $SU(3)$, which reads

$$\Gamma^\mu = 1_3 \otimes (\gamma_0, \gamma) = (1_3 \otimes \sigma_x \otimes 1_2, 1_3 \otimes i\sigma_y \otimes \sigma), \quad (31)$$

which we want to bring into the form

$$\tilde{\Gamma}^\mu = (\gamma_0, \gamma) \otimes 1_3 = (\sigma_x \otimes 1_2 \otimes 1_3, i\sigma_y \otimes \sigma \otimes 1_3). \quad (32)$$

For that purpose we define, in analogy to the previous case for $SU(2)$, the following matrices

$$Y = X \otimes 1_2, Y^\dagger = X^\dagger \otimes 1_2; \quad Z = 1_2 \otimes X, Z^\dagger = 1_2 \otimes X^\dagger, \quad (33)$$

by which we obtain for the combined transformation the concise result

$$ZY \Gamma_{0,2,3} Y^\dagger Z^\dagger = \tilde{\Gamma}_{0,2,3}, \quad ZY \Gamma_1 Y^\dagger Z^\dagger = -\tilde{\Gamma}_1, \quad (34)$$

whereby we have to carefully obey the sequence of the multiplications and use the associative law of tensor multiplication of matrices. Additionally note that YZ is not equal to ZY . Both Y and Z are fairly large 12×12 matrices. The spinor field Ψ thereby transforms as $\tilde{\Psi} = ZY\Psi$. The minus sign at the 1-component has no physical meaning, as the Clifford algebra is not affected by it, which has its origin in the transformation properties of σ_3 according to the matrix X in (29). Therefore, Γ^μ and $\tilde{\Gamma}^\mu$ are unitarily equivalent, which is the main point we wanted to show. The general phase operator of $\tilde{\Psi}$ for the $SU(3)$ symmetry group then reads

$$\tilde{P}(x) = \exp(i\theta^a(x) 1_4 \otimes G^a), \quad (35)$$

where G^a denotes a standard generator of $SU(3)$, and the index a runs from 1 to 8. The previous general considerations of Section 5 remain of course valid for the present case of $N = 3$.

7. Partial Unification of Symmetries

The intention of this last section is to include several symmetry groups of $SU(N)$ for various N in the extended Dirac equation, for example, the unified $SU(2) \otimes SU(4)$ symmetry as discussed in references [11,12], which also contain a table of the $SU(4)$ group. We concentrate below on $SU(3)$ (a subgroup of $SU(4)$), which is relevant for the strong interactions of colored quarks. The common dimension of the matrices involved then is eight ($2 \times 4 = 8$). Following the previous reasoning, we shall employ therefore the following extended Dirac equation

$$\Gamma^\mu = \gamma^\mu \otimes 1_8, \quad \Gamma_5 = \gamma_5 \otimes 1_8. \quad (36)$$

Although we will mainly consider $SU(3)$ as a subgroup of $SU(4)$, we shall retain the fifteenth element of the latter group, which commutes with all the elements of the subgroup $SU(3)$, and is represented by the diagonal 4×4 hypercharge matrix operator

$$h = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}. \quad (37)$$

It has a vanishing trace and is normalized such that $\text{Trace}(h^2) = 1$. The elements of $SU(3)$ as subgroup of $SU(4)$ are named G^a (with index a running from 1 to 8) then have the appearance of the matrix (37), yet with different normalization factor, and -3 being replaced by 0, and the remaining 3×3 unit matrix by the elements of $SU(3)$. Therefore, G^a commutes with h . We can then define the corresponding 32×32 matrices as

$$S^a = 1_4 \otimes 1_2 \otimes G^a. \quad (38)$$

The second dimensional factor is needed to accommodate commutation with the elements of the symmetry group $SU(2)$. These elements are given by the Pauli matrices in Equation (1). Consistently with the extended Dirac equation, the isospin matrices also need to be formally extended to 32×32 matrices as follows

$$\mathbf{I} = \frac{1}{2} 1_4 \otimes \sigma \otimes 1_4. \quad (39)$$

Comparison of Equations (38) and (39) with (36) shows that the gamma matrices commute with all elements of both symmetry groups. The tensor product notation makes this fact transparent. The same conclusion also holds for the hypercharge operator H , which is adequately defined as

$$H = 1_4 \otimes 1_2 \otimes h. \quad (40)$$

Collecting all terms, we can therefore write the summed up and unified symmetry-group elements following the notation of (16) and (25) in the concise exponential form

$$P(x) = \exp(i\varphi(x)1_{32}) \exp(i\lambda(x) \cdot \mathbf{I}) \exp(i\theta^a(x) S^a) \exp(i\theta(x) H). \quad (41)$$

The first exponential factor just considers a trivial phase. For local symmetries the various parameters are assumed to vary in space-time, for global they are all constant. Close inspection and comparison of the Equations (38), (39) and (36) confirms that all the group elements of $SU(2)$ and $SU(3) \oplus H$ commute with each other, and so also do all the related exponential functions that appear in (41), which can therefore also be written as a

single exponential function. In association with this general phase operator, after (5) the summed up covariant derivative (for $\varphi = 0$) is given as

$$D_\mu = \partial_\mu - i \left(g \mathbf{U}_\mu(x) \cdot \mathbf{I} + g' V_\mu^a(x) S^a + g'' B_\mu(x) H \right), \quad (42)$$

where g is the coupling constant for $SU(2)$, g' for $SU(3)$ and g'' for H . Here, the related gauge four-vector fields are $V_\mu^a(x)$ for $SU(3)$ (with a running from 1 to 8), $B_\mu(x)$ for the hypercharge H and the three-vector field $\mathbf{U}_\mu(x)$ for $SU(2)$. In this approach of unified gauge symmetry we followed the unification model suggested by Marsch and Narita [11,12].

8. Summary and Conclusions

In conclusion, we may say that various ways exist to unite $SU(N)$ symmetry with the standard Dirac equation for a spin-one-half fermion with its four (particle/antiparticle and spin up/down) degrees of freedom. Instead of transforming the Dirac spinor multiplet (7) under $SU(N)$ with the transformation (6), we may extend the Dirac equation by tensor-multiplication at the right to obtain the related gamma matrices as $\tilde{\Gamma}^\mu = \gamma^\mu \otimes 1_N$, which also have the dimension of $4N$. By repeated multiplications, we can even accommodate several symmetry groups together in a single Dirac equation. This possibility was briefly evaluated in the previous section.

Of course, the mathematical scheme discussed here can be easily extended to larger symmetry groups, many of which have been discussed in the ample literature. In particular, the group $SU(5)$ has been proposed [15] or $SO(10)$ [16], which can accommodate all fermions of the first family in a sixteen-component spinor representation. Lucid introductions into the subject of unification of gauge interactions can be found in the cited textbooks [2–4] and the recent concise textbook of Fritzsche [17] for German readership.

Finally, we want to recall that chiral symmetry is also valid for the massless extended Dirac equation, for which then the projection operator $\tilde{P}_\pm = \frac{1}{2}(1_{4N} \pm \tilde{\Gamma}_5)$ after (24) commutes as well with all the $SU(N)$ symmetry elements as defined in (23). Consequently, for $m = 0$ this symmetry applies separately to the left- and right-chiral extended Weyl spinor fields.

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