Sitzungsber. Abt. II (2005) 214: 27–41

Sitzungsberichte
Mathematisch-naturwissenschaftliche Klasse Abt. II
Mathematische, Physikalische und Technische Wissenschaften

One-Parameter Closed Dual Spherical Motions and Holditch's Theorem

By

Rashad A. Abdel-Baky

(Vorgelegt in der Sitzung der math.-nat. Klasse am 23. Juni 2005
durch das w. M. Ludwig Reich)

Abstract

In this paper, for one-parameter closed dual spherical motions, we define the dual versions of the area vector of a given closed space curve, and the area projection of this curve in the direction of a given unit vector. The relationship between the above dual versions and the dual Steiner vector of the motion is used to give a generalization for Holditch's theorem of planar kinematics into space kinematics. The geometry of ruled surfaces generated in a one-parameter spatial motion are treated in terms of their integral invariants. Finally, an example of application is investigated and explained in detail.

Mathematics Subject Classification (2000): 53A04, 53A05, 53A17.
Key words: Ruled surface, integral invariants, dual area vector, Holditch's theorem.

1. Introduction

HOLDITCH's theorem [13] of planar kinematics states: The ring area between a closed convex curve α and the curve β traced out by a point on a chord of fixed length that slides around with both endpoints on α is independent from the shape of α. It only depends on the choice of the fixed point on the chord. This theorem has been generalized to the closed ruled surfaces in Euclidean 3-space E^3 by MÜLLER [15], HOSCHEK [14], and HACISALIHOĞLU [11]. Some relations between the pitches and the angles of pitch of closed ruled surfaces can be found in [6, 15]. Important contributions to this theorem have been studied in [15–17].

As it is known, the other analytical tool in the study of three-dimensional kinematics and the differential geometry of ruled surfaces is based upon dual vector calculus as shown in [1–3, 13, 18]. Although the area vector of a closed space curve in E^3 is known, however, the dual area vector is not. So, this led us to a definition of the dual area vector of a closed spherical curve. Thus, we have composed the dual area vector and the dual Steiner vector of one-parameter dual spherical motions. Making use of this relationship a generalization of HOLDITCH's theorem to space kinematics is given. Moreover, some new relations between integral invariants of closed ruled surfaces, generated under the motion, were obtained.

Line trajectories are important in kinematic design because they can be identified with lines of kinematic elements of particular mechanism. In spatial motion, the trajectories of oriented lines embedded in a moving rigid body are generally ruled surfaces. Thus, the geometry of ruled surfaces is important in the study of rational design problems in spatial mechanisms.

An oriented line in Euclidean 3-space E^3 may be given by two points $\mathbf{x}$ and $\mathbf{y}$ on it. The parametric equation of the line is

$$\mathbf{y} = \mathbf{x} + \mu\mathbf{a}, \tag{1.1}$$

$\mathbf{a}$ is a unit vector along the line. Then we define the moment of the vector $\mathbf{a}$ with respect to a fixed origin point in E^3 as

$$\mathbf{a}^* = \mathbf{y}\times\mathbf{a} = \mathbf{x}\times\mathbf{a}. \tag{1.2}$$

This means that $\mathbf{a}^*$ is the same for all choices of the points on the line, and the pair $(\mathbf{a}, \mathbf{a}^*) \in E^3 \times E^3$ satisfies the following relations:

$$\langle\mathbf{a}, \mathbf{a}\rangle = 1, \qquad \langle\mathbf{a}, \mathbf{a}^*\rangle = 0. \tag{1.3}$$

The six components a_i, a_i^* $(i = 1, 2, 3)$ of $\mathbf{a}$, and $\mathbf{a}^*$ are called the normed Plücker coordinates of the line.

A ruled surface in Euclidean 3-space E^3 is a differentiable one-parameter set of straight lines. Such a surface has a parameterization of the form

$$\mathbf{y}(t, \mu) = \mathbf{x}(t) + \mu\mathbf{a}(t), \qquad t, \mu \in \Re, \tag{1.4}$$

where $\mathbf{x} = \mathbf{x}(t)$ is its base curve and $\mathbf{a} = \mathbf{a}(t)$ is the unit vector giving the direction of the straight lines of the surface.

Let $\{\mathbf{r};\ \mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ and $\{\mathbf{0}_f;\ \mathbf{f}_1,\ \mathbf{f}_2,\ \mathbf{f}_3\}$ be two right-handed sets of orthogonal unit vectors which are rigidly linked to the moving space H_m and the fixed space H_f, respectively. In this case, the structural equations of a one-parameter spatial motion of H_m with respect to H_f are as follows:

$$d\mathbf{r} = \sum_i \omega_i \mathbf{e}_i, \qquad d\mathbf{e}_i = \sum_j \omega_{ij} \mathbf{e}_j, \qquad 1 \le i,\ j \le 3, \tag{1.5}$$

where $\omega_{ij}(t)$ and $\omega_i(t)$ are the differential forms of the motion and $t \in \Re$. We shall denote the one-parameter spatial motion by H_m/H_f.

During the one-parameter spatial motion H_m/H_f, let the set $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ complete a spatial motion along a closed curve $\mathbf{r}(t)$. Then the $\mathbf{e}_1$-axis generates a closed ruled surface. We can describe the surface by the equation

$$R: \quad \mathbf{y}(t,\mu) = \mathbf{r}(t) + \mu \mathbf{e}_1(t), \qquad t, \mu \in \Re. \tag{1.6}$$

For this ruled surface

$$\mathbf{y}(t,\mu) = \mathbf{y}(t + 2\pi, \mu) \tag{1.7}$$

can be taken as a condition for being closed. The reason for choosing the $\mathbf{e}_1$-axis is for the sake of simplicity.

The pitch of R is defined by

$$L_{e_1} = \oint d\mu = -\oint \langle d\mathbf{r}, \mathbf{e}_1 \rangle. \tag{1.8}$$

An orthogonal trajectory of R starting from the point p_0 on an $\mathbf{e}_1$-generator intersects the same generator at another point p_1 which is generally different from p_0, i.e. $L_{e_1} = p_0 p_1$.

Let us choose a unit vector

$$\mathbf{n} = \cos\phi\, \mathbf{e}_2 + \sin\phi\, \mathbf{e}_3. \tag{1.9}$$

It is clear that

$$d\phi = -\langle d\mathbf{e}_2, \mathbf{e}_3 \rangle = \langle d\mathbf{e}_3, \mathbf{e}_2 \rangle. \tag{1.10}$$

The total of ϕ is called the angle of pitch of R and is given by

$$\lambda_{e_1} = \oint d\phi. \tag{1.11}$$

The pitch and the angle of pitch are well-known real integral invariants of a closed ruled surface [1, 6, 11, 12].

The area vector of a closed space curve $\mathbf{x} = \mathbf{x}(t)$ in Euclidean 3-space E^3 is given by

$$\mathbf{a}_x = \oint \mathbf{x} \times d\mathbf{x}, \tag{1.12}$$

and the projection area of this curve in the direction of a unit vector $\mathbf{e}$ normal to the projection plane is given as (see [15]):

$$2\sigma_{xe} = \langle \mathbf{a}_x, \mathbf{e} \rangle. \tag{1.13}$$

2. Dual Spherical Motions

A dual number A has the form $a + \varepsilon a^*$, where a, a^* are real numbers. Here ε is a dual unit subject to the rules $\varepsilon \neq 0$, $\varepsilon^2 = 0$, $\varepsilon \cdot 1 = 1 \cdot \varepsilon = \varepsilon$. The set of dual numbers D forms a commutative ring having the numbers εa^* (a^* real) as divisors of zero. D is not a field. No number εa^* has an inverse in the algebra. But the other laws of the algebra of dual numbers are the same as of the complex numbers.

For all pairs $(\mathbf{a}, \mathbf{a}^*) \in E^3 \times E^3$ the set

$$D^3 = \{\mathbf{A} = \mathbf{a} + \varepsilon \mathbf{a}^*,\ \varepsilon \neq 0,\ \varepsilon^2 = 0\}, \tag{2.1}$$

together with the scalar product

$$\langle \mathbf{A}, \mathbf{B} \rangle = \langle \mathbf{a}, \mathbf{b} \rangle + \varepsilon(\langle \mathbf{b}, \mathbf{a}^* \rangle + \langle \mathbf{b}^*, \mathbf{a} \rangle), \tag{2.2}$$

forms the dual 3-space D^3. Thereby a point $\mathbf{A} = (A_1, A_2, A_3)^t$ has dual coordinates $A_i = (a_i + \varepsilon a_i^*) \in D$. The norm is defined by

$$\langle \mathbf{A}, \mathbf{A} \rangle^{1/2} =: \|\mathbf{A}\| = \|\mathbf{a}\| \left(1 + \varepsilon \frac{\langle \mathbf{a}, \mathbf{a}^* \rangle}{\|\mathbf{a}\|^2} \right). \tag{2.3}$$

In the dual 3-space D^3 the dual unit sphere is defined by

$$K = \{\mathbf{A} \in D^3 \mid \|\mathbf{A}\|^2 = A_1^2 + A_2^2 + A_3^2 = 1\}. \tag{2.4}$$

2.1. E. Study's Map

The set of all oriented lines in Euclidean 3-space E^3 are in one-to-one correspondence with the set of the points of the dual unit sphere in the dual 3-space D^3 [4].

The E. Study map allows us to rewrite Eq. (1.6) by the dual vector function as

$$R: \quad \mathbf{E}_1(t) = \mathbf{e}_1(t) + \varepsilon \mathbf{r}(t) \times \mathbf{e}_1(t), \tag{2.5}$$

since the spherical image $\mathbf{e}_1$ is a unit vector, the dual vector $\mathbf{E}_1$ also has unit length as is seen from the computation

$$\begin{aligned} \langle \mathbf{E}_1, \mathbf{E}_1 \rangle &= \langle \mathbf{e}_1 + \varepsilon \mathbf{r} \times \mathbf{e}_1, \mathbf{e}_1 + \varepsilon \mathbf{r} \times \mathbf{e}_1 \rangle \\ &= \langle \mathbf{e}_1, \mathbf{e}_1 \rangle + 2\varepsilon \langle \mathbf{e}_1, \mathbf{r} \times \mathbf{e}_1 \rangle + \varepsilon^2 \langle \mathbf{r} \times \mathbf{e}_1, \mathbf{r} \times \mathbf{e}_1 \rangle = \langle \mathbf{e}_1, \mathbf{e}_1 \rangle = 1. \end{aligned} \tag{2.6}$$

The differentiable curve

$$t \in \Re \rightarrow \mathbf{E}_1(t) \in K \tag{2.7}$$

represents a differentiable family of straight lines of Euclidean 3-space E^3. The lines $\mathbf{E}_1(t)$ are the generators of a surface. Hence, ruled surfaces and dual curves are synonymous in this paper.

Suppose that the dual frame $\{F\}$ (called the fixed dual frame), which is composed of three mutually orthogonal oriented lines $\{\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3\}$, is rigidly attached to the fixed space H_f. Also suppose that the dual frame $\{E\}$ (called the moving dual frame), which is composed of three mutually orthogonal oriented lines $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, is rigidly attached to the moving space H_m. The oriented lines $\mathbf{E}_i$ and $\mathbf{F}_i$ are given by

$$\mathbf{E}_i = \mathbf{e}_i + \varepsilon \mathbf{e}_i^* \qquad \textit{and} \qquad \mathbf{F}_i = \mathbf{f}_i + \varepsilon \mathbf{f}_i^* \qquad (i = 1,\ 2,\ 3), \tag{2.8}$$

where

$$\mathbf{e}_i^* = \mathbf{r} \times \mathbf{e}_i \qquad \textit{and} \qquad \mathbf{f}_i^* = \mathbf{00}_f \times \mathbf{e}_i, \tag{2.9}$$

in which $\mathbf{0}$ is a fixed point as origin of E^3. VELDKAMP [18] assumed that both of these frames are attached to separate dual unit spheres K_f and K_m with the same center $\mathbf{O}$ in the dual 3-space D^3. Then any point on the dual unit sphere can be written unambiguously as a linear combination of $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ as well as of $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$. We have therefore for a point X:

$$X_1\mathbf{E}_1 + X_2\mathbf{E}_2 + X_3\mathbf{E}_3 = \widetilde{X}_1\mathbf{F}_1 + \widetilde{X}_2\mathbf{F}_2 + \widetilde{X}_3\mathbf{F}_3. \tag{2.10}$$

The column vectors

$$\widetilde{\mathbf{X}} = \begin{pmatrix} \widetilde{X}_1 \\ \widetilde{X}_2 \\ \widetilde{X}_3 \end{pmatrix} \qquad \textit{and} \qquad \mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \tag{2.11}$$

are the position vectors of X with respect to K_m and K_f, respectively. We derive from (2.10)

$$X_i = \sum_{j=1}^{3} \langle \mathbf{E}_i, \mathbf{F}_j \rangle \widetilde{X}_j \qquad (i = 1,\ 2,\ 3). \tag{2.12}$$

Putting $\langle \mathbf{E}_i, \mathbf{F}_j \rangle = A_{ij}$ and introducing the matrix $A = (A_{ij})$, we see that (2.12) expresses that

$$\mathbf{X} = A\widetilde{\mathbf{X}}. \tag{2.13}$$

Since $\mathbf{F}_j$ is a linear combination of $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$, we put $\mathbf{F}_j = B_{1j}\mathbf{E}_1 + B_{2j}\mathbf{E}_2 + B_{3j}\mathbf{E}_3$; then $B_{ij} = \langle \mathbf{E}_i, \mathbf{F}_j \rangle = A_{ij}$. Therefore $\mathbf{F}_j = A_{1j}\mathbf{E}_1 + A_{2j}\mathbf{E}_2 + A_{3j}\mathbf{E}_3$. Hence

$$\delta_{ij} = \langle \mathbf{E}_i, \mathbf{E}_j \rangle = \langle \mathbf{F}_i, \mathbf{F}_j \rangle, \tag{2.14}$$

where δ_{ij} is the Kronecker symbol. This shows that A is an orthogonal dual matrix. Then we say that $\{\mathbf{O};\ \mathbf{E}_1,\ \mathbf{E}_2,\ \mathbf{E}_3\}$ moves with respect to $\{\mathbf{O};\ \mathbf{F}_1,\ \mathbf{F}_2,\ \mathbf{F}_3\}$. We may interpret this as follows: The dual unit sphere K_m rigidly connected with $\{\mathbf{O};\ \mathbf{E}_1,\ \mathbf{E}_2,\ \mathbf{E}_3\}$ moves over the dual unit sphere K_f rigidly connected with $\{\mathbf{O};\ \mathbf{F}_1,\ \mathbf{F}_2,\ \mathbf{F}_3\}$. This motion is called a one-parameter dual spherical motion and will be denoted by K_m/K_f.

Theorem 2.1. *The Euclidean motions in* E^3 *are represented in* D^3 (*the dual space*) *by dual orthogonal* 3×3 *matrices* $A = (A_{ij})$, *where* $AA^t = I$, A_{ij} *are dual numbers*, *and* I *is the* 3×3 *unit matrix.*

According to Theorem 2.1 the 3×3 dual matrix $A(t)$ of the motion K_m/K_f represents the one-parameter spatial motion H_m/H_f with the same parameter $t \in \Re$. If the matrix $A(t)$ is a periodic function, i.e. $A(t) = A(t + 2\pi)$, the motion K_m/K_f (hence also H_m/H_f) is called a closed motion, otherwise it is called an open motion.

The Lie algebra $L(O_{D^3})$ of the group GL of 3×3 positive orthogonal dual matrices A is the algebra of skew-symmetric 3×3 dual matrices

$$\Omega = dAA^t = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix}, \tag{2.15}$$

where dA indicates the differentiation of A with respect to the real parameter t. During the motion K_m/K_f the differential velocity vector of a fixed dual point X on K_m, analogous to the real spherical motion [10], is

$$d\mathbf{X} = \mathbf{\Omega} \times \mathbf{X}, \tag{2.16}$$

where $\mathbf{\Omega} = \boldsymbol{\omega} + \varepsilon\boldsymbol{\omega}^*$ is called the instantaneous rotation vector of the motion K_m/K_f; $\boldsymbol{\omega}$ and $\boldsymbol{\omega}^*$, respectively corresponding to the instantaneous rotational differential velocity vector and the instantaneous translational differential velocity vector of the corresponding spatial motion H_m/H_f.

The instantaneous dual spherical centers of rotation are called the dual poles $\mathbf{P}_m$ and $\mathbf{P}_f$ on K_m and K_f. During the motion K_m/K_f, the trajectories of these points form closed pole curves. The dual Steiner vector of the closed motion K_m/K_f is given by

$$\mathbf{S} = \mathbf{s} + \varepsilon \mathbf{s}^* = \oint \mathbf{\Omega}, \tag{2.17}$$

where the integration is taken along the pole curve (P).

We need the following theorem:

Theorem 2.2. *For a one-parameter motion K_m/K_f a closed curve (X) on K_f of an arbitrary fixed point $\mathbf{X}$ on K_m, the following holds* ([6], [12]):

(i) The dual angle of pitch Λ_x of the closed ruled surface generated by $\mathbf{X} = \mathbf{X}(t)$ is equal to the projection of the generator onto the Steiner vector $\mathbf{S}$ of the motion, that is

$$\Lambda_x = \lambda_x - \varepsilon L_x = \langle \mathbf{X}, \mathbf{S} \rangle. \tag{2.18}$$

(ii) The dual spherical area enclosed by (X) may be calculated by

$$F_x = f_x + \varepsilon f_x^* = 2\pi(1 - n) - \Lambda_x, \tag{2.19}$$

where n is the (integer real) number of rotations of the pole curve (P) at $\mathbf{X}$.

3. A Generalization of the Holditch's Theorem

For the one-parameter motion K_m/K_f, let (X) be a closed dual curve on K_f of an arbitrary fixed point $\mathbf{X}$ of K_m. Then, as in Eq. (1.12), the dual area vector of (X) is given by

$$\mathbf{A}_x = \oint \mathbf{X} \times d\mathbf{X}. \tag{3.1}$$

According to Eq. (2.16) and Theorem 2.2 we have:

$$\begin{aligned} \mathbf{A}_x &:= \oint \mathbf{X} \times (\mathbf{\Omega} \times \mathbf{X}) = \oint [\langle \mathbf{X}, \mathbf{X} \rangle \mathbf{\Omega} - \langle \mathbf{X}, \mathbf{\Omega} \rangle \mathbf{X}] \\ &= \oint \mathbf{\Omega} - \left\langle \mathbf{X}, \oint \mathbf{\Omega} \right\rangle \mathbf{X} = \mathbf{S} - \langle \mathbf{X}, \mathbf{S} \rangle \mathbf{X} \\ &= \mathbf{S} - \Lambda_x \mathbf{X}. \end{aligned} \tag{3.2}$$

Theorem 3.1. *For the motion K_m/K_f let (X) be a closed dual curve on K_f of an arbitrary fixed point $\mathbf{X}$ of K_m. The area vector $\mathbf{A}_x$ drawn by $\mathbf{X}$ may be calculated by*

$$\mathbf{A}_x = \mathbf{S} - \Lambda_x \mathbf{X}. \tag{3.3}$$

As a result the following corollary can be given:

Corollary 3.1. *The dual area vector of the moving pole curve (P) of the one-parameter closed dual spherical motion K_m/K_f is the zero vector.*

From the relation (3.3) we can obtain that

$$\|\mathbf{A}_x\|^2 = \|\mathbf{S}\|^2 - \Lambda_x^2. \tag{3.4}$$

If $\Theta = \theta + \varepsilon\theta^*$ is the dual angle between $\mathbf{S}$ and $\mathbf{A}_x$, then we may write

$$\langle \mathbf{A}_x, \mathbf{S}\rangle = \|\mathbf{A}_x\|\|\mathbf{S}\|\cos\Theta. \tag{3.5}$$

From Eqs. (3.3)–(3.5), we get

$$\cos\Theta = \frac{\sqrt{\|\mathbf{S}\|^2 - \Lambda_x^2}}{\|\mathbf{S}\|}. \tag{3.6}$$

On the other hand, if $\Phi = \varphi + \varepsilon\varphi^*$ is the dual angle between $\mathbf{X}$ and $\mathbf{S}$, then

$$\langle \mathbf{X}, \mathbf{S}\rangle = \Lambda_x = \|\mathbf{S}\|\cos\Phi. \tag{3.7}$$

We have found, by substituting Eq. (3.7) into Eq. (3.6), that

$$\cos\Theta = \sin\Phi. \tag{3.8}$$

Hence the following theorem can be given:

Theorem 3.2. *For the motion K_m/K_f, a dual unit vector $\mathbf{X}$ and the dual unit vector along its dual area vector $\mathbf{A}_x$ may be interchanged, leaving the dual Steiner vector of the motion invariant.*

If we calculate the real and dual parts of Eq. (3.8), we have

$$\varphi = \frac{\pi}{2} + \theta, \qquad \varphi^* = -\theta^*. \tag{3.9}$$

Applying to Study's map, we obtain as a result the following

Theorem 3.3. *For the one-parameter closed spatial motion H_m/H_f, an oriented line and the line along its area vector are at equal minimal distance from the Steiner vector of the motion, and the angles between the Steiner vector and these lines are complementary angles.*

Taking the scalar product of formula (3.3) with Steiner's vector, the result is

$$\langle \mathbf{A}_x, \mathbf{S}\rangle = \langle \mathbf{S}, \mathbf{S}\rangle - \Lambda_x\langle \mathbf{S}, \mathbf{X}\rangle \Leftrightarrow \|\mathbf{A}_x\|\left\langle \frac{\mathbf{A}_x}{\|\mathbf{A}_x\|}, \mathbf{S}\right\rangle = \|\mathbf{S}\|^2 - \Lambda_x^2 \tag{3.10}$$

or equivalently

$$\Lambda_{a_x}\|\mathbf{A}_x\| + \Lambda_x^2 = \|\mathbf{S}\|^2, \tag{3.11}$$

where Λ_{a_x} is the dual angle of pitch of the ruled surface generated by the line $\mathbf{A}_x/\|\mathbf{A}_x\|$. From Eqs. (3.4) and (3.11), we get

$$\Lambda_{a_x} = \sqrt{\|\mathbf{S}\|^2 - \Lambda_x^2} \quad \Leftrightarrow \quad \Lambda_{a_x}^2 + \Lambda_x^2 = \|\mathbf{S}\|^2. \tag{3.12}$$

Now, we are ready to give a generalization of Holditch's theorem. For this purpose, for the motion K_m/K_f, let us take a dual arc segment $\widehat{MN}$ with constant length on a great circle of K_m. If endpoints M and N of the dual arc segment $\widehat{MN}$ lie on the curve (X) on K_f, we have a special spherical motion. Denote the dual angles of $\widehat{MX}$, $\widehat{XN}$ and $\widehat{MN}$, respectively, by

$$\Phi_i = \varphi_i + \varepsilon\varphi_i^*, \qquad \Phi = \varphi + \varepsilon\varphi^* \quad (i = 1, 2), \qquad \Phi = \Phi_1 + \Phi_2. \tag{3.13}$$

Then we have

$$\mathbf{X} = \frac{\mathbf{M}\sin\Phi_1 + \mathbf{N}\sin\Phi_2}{\sin\Phi}. \tag{3.14}$$

On the other hand, by applying formula (3.3) to Eq. (3.14) we obtain

$$\begin{aligned}\mathbf{A}_x &:= \mathbf{S} - \left\langle \mathbf{S}, \left(\frac{\mathbf{M}\sin\Phi_1 + \mathbf{N}\sin\Phi_2}{\sin\Phi}\right)\right\rangle \mathbf{X} \\ &= \mathbf{S} - \left(\frac{\Lambda_m \sin\Phi_1 + \Lambda_n \sin\Phi_2}{\sin\Phi}\right)\mathbf{X} = \mathbf{S} - \Lambda_x\mathbf{X}.\end{aligned} \tag{3.15}$$

Taking the scalar product of the last relation with the dual unit vector $\mathbf{X}$, we get

$$\Lambda_x = \frac{\Lambda_m \sin\Phi_1 + \Lambda_n \sin\Phi_2}{\sin\Phi}. \tag{3.16}$$

Since the dual points M and N draw the same dual curve, that is, generate the same ruled surface, we can take $\Lambda_m = \Lambda_n$. Then the last formula yields

$$\frac{\Lambda_x}{\Lambda_m} = \frac{\sin\Phi_1 + \sin\Phi_2}{\sin\Phi}, \tag{3.17}$$

which shows that the ratio does not depend on the shape of the curve (M) on K_f. It depends only on the choice of the point X on the dual arc segment $\widehat{MN}$. This means that it does not depend on the motion

K_m/K_f. This is a generalization of Holditch's theorem of planar kinematics to the one-parameter closed dual spherical motion K_m/K_f.

Using Study's map, if we calculate the real and dual parts of this formula, we get:

$$\frac{\lambda_x}{\lambda_m} = \frac{\sin\varphi_1 + \sin\varphi_2}{\sin\varphi},$$
$$L_x = \frac{\varphi^*\lambda_x + (\sin\varphi_1 + \sin\varphi_2)L_m - \lambda_m(\varphi_1^*\cos\varphi_1 + \varphi_2^*\cos\varphi_2)}{\sin\varphi}. \tag{3.18}$$

This is a generalization of Holditch's theorem of planar kinematics to the one-parameter closed spatial motion H_m/H_f. Moreover, making use of the last formulae all the results in [6, 7, 11, 12] can be obtained.

As special case of formula (3.3), let $\mathbf{E}_1(t)$ be a dual curve on K. As usual Blaschke's frame relative to $\mathbf{E}_1$ will be defined as the frame of which this line and the central normal $\mathbf{E}_2$ to the ruled surface at the central point of $\mathbf{E}_1$ are two edges. The third edge $\mathbf{E}_3$ is the central tangent to the ruled surface $\mathbf{E}_1(t)$. The frame $\{\mathbf{E}_1 = \mathbf{E}_1(t), \mathbf{E}_2(t) = \mathbf{E}_1'/\|\mathbf{E}_1'\|, \mathbf{E}_3(t) = \mathbf{E}_1 \times \mathbf{E}_2\}$ is called Blaschke's frame. During the one-parameter spatial motion H_m/H_f the corresponding lines intersect at the striction point of the ruled surface $\mathbf{E}_1 = \mathbf{E}_1(t)$. Therefore, the structural equation of the dual spherical motion K_m/K_f is given by

$$\frac{d}{dt}\begin{pmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{pmatrix} = \begin{pmatrix}0 & P & 0\\ -P & 0 & Q\\ 0 & -Q & 0\end{pmatrix}\begin{pmatrix}\mathbf{E}_1\\ \mathbf{E}_2\\ \mathbf{E}_3\end{pmatrix}, \tag{3.19}$$

where the dual functions $P = p + \varepsilon p^* = \|\mathbf{E}_1'\|$, and $Q = q + \varepsilon q^* = \det(\mathbf{E}_1, \mathbf{E}_1', \mathbf{E}_1'')/\|\mathbf{E}_1'\|^2$ are called the Blaschke invariants of the ruled surface $\mathbf{E}_1 = \mathbf{E}_1(t)$. On the other hand, the Steiner vector will be

$$\mathbf{S} = \left(\oint Q\right)\mathbf{E}_1 + \left(\oint P\right)\mathbf{E}_3 = \Lambda_{e_1}\mathbf{E}_1 + \Lambda_{e_3}\mathbf{E}_3, \tag{3.20}$$

where Λ_{e_1} and Λ_{e_3} are the dual angles of the ruled surfaces $\mathbf{E} = \mathbf{E}_1(t)$ and $\mathbf{E}_3 = \mathbf{E}_3(t)$, respectively.

To examine the dual spherical area drawn by the second axis of the moving Blaschke frame from Eq. (3.20) we state that: The dual angle of pitch of closed ruled surfaces generated by the $\mathbf{E}_2$-axis is always zero. Thus, in view of Eq. (2.19), we have the following theorem:

Theorem 3.4. *For the one-parameter closed dual spherical motion of Blaschke's frame, the dual area vector of the second axis is parallel* *to the Steiner vector of the motion. Moreover, the spherical indicatrix of the second axis divides the unit spherical surface area into two equal parts, i.e.* $f_{e_2} = 2\pi$.

From Eqs. (3.12) and (3.21) we obtain

$$\Lambda_{a_x}^2 + \Lambda_x^2 = \Lambda_{e_1}^2 + \Lambda_{e_3}^2. \tag{3.21}$$

If we calculate the real and dual parts of this equation, we get the following relations

$$\begin{aligned} \lambda_{a_x}^2 + \lambda_x^2 &= \lambda_{e_1}^2 + \lambda_{e_3}^2, \\ \lambda_{a_x} L_{ax} + \lambda_x L_x &= \lambda_{e_1} L_{e_1} + \lambda_{e_3} L_{e_3}, \end{aligned} \tag{3.22}$$

between the angles of pitch and the pitches of the ruled surfaces generated by the lines $\mathbf{A}_x / \|\mathbf{A}_x\|$, $\mathbf{X}$, $\mathbf{E}_1$ and $\mathbf{E}_3$.

With the aid of formula (3.3) and Eq. (3.21) the dual area vectors of the lines $\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3$ are

$$\mathbf{A}_{e_1} = \Lambda_{e_3} \mathbf{E}_3, \qquad \mathbf{A}_{e_2} = \mathbf{S}, \qquad \mathbf{A}_{e_3} = \Lambda_{e_1} \mathbf{E}_1. \tag{3.23}$$

By applying Theorem 2.2 in these equalities, we obtain

$$\Lambda_{a_{e_1}} = \pm \Lambda_{e_3}, \qquad \Lambda_{a_{e_2}} = \pm \sqrt{\Lambda_{e_1}^2 + \Lambda_{e_3}^2}, \qquad \Lambda_{a_{e_3}} = \pm \Lambda_{e_1}. \tag{3.24}$$

It follows that

$$\Lambda_{a_{e_2}}^2 = \Lambda_{a_{e_1}}^2 + \Lambda_{a_{e_3}}^2. \tag{3.25}$$

Then we may state the following theorem:

Theorem 3.5. *For the motion* K_m/K_f *Eqs.* (3.23)–(3.26) *hold true between integral invariants of closed ruled surfaces generated by the lines of Blaschke's frame and the corresponding ruled surfaces generated by lines of their area vectors.*

For the motion K_m/K_f for any two different fixed points $X \neq Y$, while $\mathbf{X}$ draws a closed curve (X) on K_f, the corresponding line $\mathbf{X} \in H_m$ will generate a closed ruled surface $(X) \neq (Y)$ in H_f. Then, in view of Eq. (1.12), the projection area of (X) in the $\mathbf{Y}$-direction is

$$\begin{aligned} 2\Sigma_{xy} &= 2(\sigma_{xy} + \varepsilon \sigma_{xy}^*) := \langle \mathbf{S} - \Lambda_x \mathbf{X}, \mathbf{Y} \rangle \\ &= \Lambda_y - \Lambda_x \langle \mathbf{X}, \mathbf{Y} \rangle \\ &= \Lambda_y - \Lambda_x \cos \Delta, \end{aligned} \tag{3.26}$$

where $\Delta = \delta + \varepsilon \delta^*$ is the dual angle between the dual unit vectors $\mathbf{X}$ and $\mathbf{Y}$. In (3.26) $\Sigma_{x,y}$ is given in terms of the dual angles of pitch of the closed ruled surfaces $(X) \neq (Y)$, thus Σ_{xy} is an invariant of the surface.

In fact, from (3.26), we may have

$$2\Sigma_{yx} = \Lambda_x - \Lambda_y \cos\Delta. \tag{3.27}$$

It follows from Eqs. (3.26) and (3.27) that

$$2(\Lambda_x\Sigma_{yx} - \Lambda_y\Sigma_{xy}) = \Lambda_x^2 - \Lambda_y^2, \tag{3.28}$$

which is a relation between the integral invariants of the ruled surfaces $(X) \neq (Y)$.

In view of Eqs. (3.4) and (3.24) we can find the projection areas of the $\mathbf{E}_i$-axis of the Blaschke frame in the direction of the $\mathbf{E}_j$-axis as follows:

$$\begin{aligned} 2\Sigma_{e_ie_j} &= \langle \mathbf{A}_{e_i}, \mathbf{E}_j \rangle \\ &= \langle \mathbf{S} - \Lambda_{ei}\mathbf{E}_i, \mathbf{E}_j \rangle \\ &= \Lambda_{e_j} - \Lambda_{e_i}\delta_{ij}, \end{aligned} \tag{3.29}$$

or in a compact form:

$$2\Sigma_{e_ie_j} = \begin{cases} 0, & i = j \\ \Lambda_{e_j}, & i \neq j. \end{cases} \tag{3.30}$$

4. An Example and Remarks

For the one-parameter dual spherical motion K_m/K_f, let

$$C = \{\mathbf{X} | \langle \mathbf{X}, \mathbf{F}_1 \rangle = \text{const.}, \mathbf{X} \in K_m\} \tag{4.1}$$

be a dual curve on K_f. Thus, the dual unit vector $\mathbf{X}$ can be expressed as

$$\mathbf{X} = \cos\Theta\mathbf{F}_1 + \sin\Theta\cos\Phi\mathbf{F}_2 + \sin\Theta\sin\Phi\mathbf{F}_3, \tag{4.2}$$

where

$$\Theta = \theta + \varepsilon\theta^*, \qquad \Phi = \varphi + \varepsilon\varphi^*. \tag{4.3}$$

This means that

$$\theta = c_1(\text{real const.}), \qquad \theta^* = c_2(\text{real const.}). \tag{4.4}$$

Thus Eq. (4.2) has only two real parameters φ and φ^*. So, if we choose $\varphi^* = h\varphi$, h denoting to the pitch of the motion H_m/H_f, and φ as the motion parameter, then Eq. (4.2) represents a ruled surface in

H_f-space. Thus, Blaschke's frame of the ruled surface $\mathbf{X} = \mathbf{X}(\varphi)$ is found as

$$\mathbf{E}_1(\varphi) = (\cos\Theta, \sin\Theta\cos\Phi, \sin\Theta\sin\Phi),$$
$$\mathbf{E}_2(\varphi) = \frac{\mathbf{E}_1'}{\|\mathbf{E}_1'\|} = (0, -\sin\Phi, \cos\Phi),$$
$$\mathbf{E}_3(\varphi) = \mathbf{E}_1 \times \mathbf{E}_2 = (\sin\Theta, -\cos\Theta\cos\Phi, -\cos\Theta\sin\Phi). \tag{4.5}$$

If we differentiate these expressions, we get

$$P = (1+\varepsilon h)\sin\Theta, \qquad Q = (1+\varepsilon h)\cos\Theta. \tag{4.6}$$

Consequently, the dual Steiner vector will be

$$\mathbf{S} = 2\pi(1+\varepsilon h)(\cos\Theta\mathbf{E}_1 + \sin\Theta\mathbf{E}_3). \tag{4.7}$$

And, according to formula (3.3), we obtain

$$\mathbf{A}_{e_1} = 2\pi(1+\varepsilon h)\sin\Theta\mathbf{E}_3,$$
$$\mathbf{A}_{e_3} = 2\pi(1+\varepsilon h)\cos\Theta\mathbf{E}_1,$$
$$\mathbf{A}_{e_2} = 2\pi(1+\varepsilon h)(\cos\Theta\mathbf{E}_1 + \sin\Theta\mathbf{E}_3). \tag{4.8}$$

On the other hand, the corresponding equations to (3.23)–(3.25) are:

$$\Lambda_{a_{e_1}} = \Lambda_{e_3} = 2\pi(1+\varepsilon h)\sin\Theta,$$
$$\Lambda_{a_{e_3}} = \Lambda_{e_1} = 2\pi(1+\varepsilon h)\cos\Theta,$$
$$\Lambda_{a_{e_2}} = \Lambda_{e_3} = 2\pi(1+\varepsilon h), \qquad \Lambda_{e_2} = 0. \tag{4.9}$$

Finally, we get

$$\Sigma_{e_1e_2} = \Sigma_{e_3e_2} = 0, \qquad \Sigma_{e_2e_1} = \Sigma_{e_3e_1} = \pi(1+\varepsilon h)\cos\Theta,$$
$$\Sigma_{e_1e_3} = \Sigma_{e_2e_3} = \pi(1+\varepsilon h)\sin\Theta. \tag{4.10}$$

Now we may calculate the equation of the ruled surface $\mathbf{X} = \mathbf{X}(\varphi)$ in terms of the Plücker coordinates. Let $\mathbf{L}$ denote a point on this surface. We can write

$$\mathbf{L}(\varphi,\mu) = \mathbf{x}(\varphi)\times\mathbf{x}^*(\varphi) + \mu\mathbf{x}(\varphi), \qquad \mu\in\Re. \tag{4.11}$$

If (l_1, l_2, l_3) are the coordinates of $\mathbf{L}$, then Eqs. (4.2) and (4.11) yield

$$l_1 = \varphi^*\sin^2\vartheta + \mu\cos\vartheta,$$
$$l_2 = -\vartheta^*\sin\varphi - (\varphi^*\cos\vartheta - \mu)\sin\vartheta\cos\varphi,$$
$$l_3 = \vartheta^*\cos\varphi - (\varphi^*\cos\vartheta - \mu)\sin\vartheta\sin\varphi. \tag{4.12}$$

The graph of the ruled surface given by Eq. (4.12) is shown in Fig. 1.

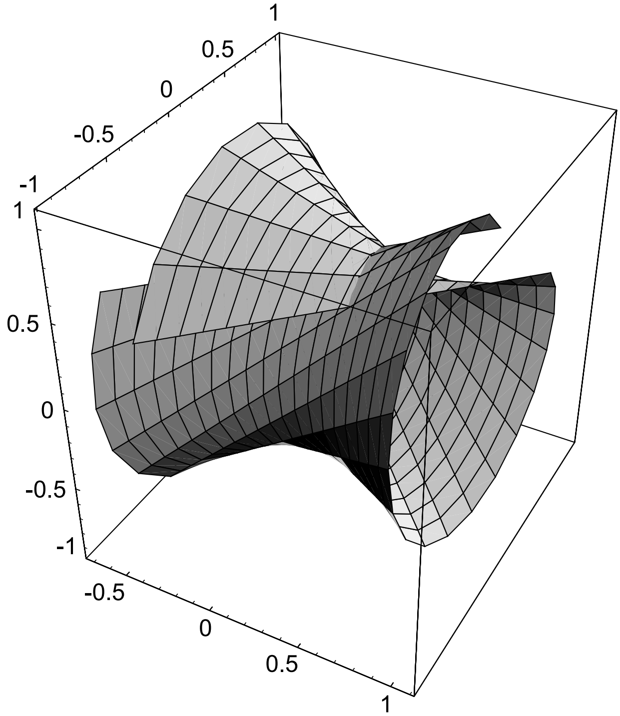

Fig. 1. Ruled surface in the domain $\varphi \in [0, 2\pi], \mu \in [-1, 1]$, $\theta = \pi/4, \theta^* = \frac{1}{2}$, $h = \frac{1}{12}$

By separating the real and dual parts of (4.8)–(4.10), we can find many relations between the angles of pitch and the pitches of ruled surfaces generated by the lines $\mathbf{E}_i$ $(i = 1, 2, 3)$ and the corresponding oriented lines along their area vectors.

5. Conclusion

The starting point of this paper is to define the dual versions of the area vector of a given closed space curve, and the area projection of this curve in the direction of a unit vector given in [15]. Introducing relationships between these quantities and the dual angle of pitch of a closed ruled surface, we give a generalization of Holditch's theorem and the geometry of ruled surfaces generated in a one-parameter spatial motion is treated in terms of their integral invariants.

References

[1] ABDEL-BAKY, R. A. (1997) The relation among Darboux vectors of ruled surfaces in a line congruence. Riv. Mat. Univ. Parma **6**: 201–211

[2] ABDEL-BAKY, R. A. (2002) An explicit characterization of dual spherical curves. Commun. Fac. Sci. Univ. Ank. Ser. A **51**: 1–9

[3] ABDEL-BAKY, R. A. (2003) On the Blaschke approach of ruled surfaces. Tamkang J. Math. **34**: 107–116

[4] GUGENHEIMER, H. W. (1956) Differential Geometry, pp. 162–169. McGraw-Hill, New York

[5] EROL, K., KELES, S. (1994) On Holditch's theorem and polar inertia momentum. Commun. Fac. Sci. Univ. Ank. Ser. A **43**: 41–47

[6] GURSOY, O. (1990) The dual angle of pitch of a closed ruled surface. Mech. Mach. Theory **25**: 131–140

[7] GURSOY, O., KUCUK, A. (1999) On the invariants of trajectory surfaces. Mech. Mach. Theory **34**: 587–597

[8] GUNES, R., KELES, S., HACISALIHOĞLU, H. H. (1989) Two corollaries of Holditch's theorem for one-parameter closed spatial motions. Commun. Fac. Sci. Univ. Ank. Ser. A **38**: 53–58

[9] GUNES, R., KELES, S., HACISALIHOĞLU, H. H. (1989) Two corollaries of Holditch's theorem for the parts of surface in E^3. Commun. Fac. Sci. Univ. Ank. Ser. A **38**: 59–66

[10] HACISALIHOĞLU, H. H. (1971) On closed spherical motions. Quart. Appl. Math.: 269–275

[11] HACISALIHOĞLU, H. H. (1972) On the pitch of a closed ruled surface. Mech. Mach. Theory **7**: 291–305

[12] HACISALIHOĞLU, H. H., ABDEL-BAKY, R. A. (1997) Holditch's theorem for one-parameter closed motions. Mech. Mach. Theory **32**: 235–239

[13] HOLDITCH, H. (1858) Geometrical theorem. Q. J. Pure Appl. Math. **2**

[14] HOSCHEK, J. (1973) Integralinvarianten von Regelflächen. Arch. Math. **XXIV**: 218–224

[15] MÜLLER, H. R. (1980) Erweiterung des Satzes von Holditch für geschlossene Raumkurven. Abhandl. Barun. Wiss. Ges. **31**: 129–135

[16] POTTMANN, H. (1985) Holditch-Sicheln. Arch. Math. **44**: 373–378

[17] POTTMANN, H., ROESCHEL, O. (1988) Öffnungsinvarianten der Bahnregelflächen euklidischer Zwangsläufe. Beitr. Alg. Geom. **27**: 27–111

[18] VELDKAMP, G. R. (1976) On the use of dual numbers, vectors, and matrices in instantaneous spatial kinematics. Mech. Mach. Theory **11**: 141–156

Author's address: Prof. Rashad A. Abdel-Baky, Department of Mathematics, Faculty of Science, University of Assiut, 71516 Assiut, Egypt. *Present address:* P.O. Box 4911, Makkah, Saudi Arabia. E-Mail: rbaky@hotmail.com.