SLOW MAGNETIC ROTATOR IN A COLLISIONLESS PLASMA. TOWARDS THE THEORY OF MAGNETOSPHERES

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Abstract

In order to study in greater detail the formation process of planetary magnetospheres, as well as electromagnetic effects connected with the motion of satellites in the interplanetary space, the problem of generation of an inductive electromagnetic field by a slowly rotating ($\Omega \ll \omega_{pe} V_e/c$) oblique magnetic dipole was solved within the frame of plasma kinetic theory. We considered the magnetic moment, μ_0, of the dipole to be distributed on the scale r_0 by Gaussian law. An analytical investigation of the asymptotic behavior of vector potential field **A** was carried out. The structure of magnetic field lines in the equatorial plane of the rotator was studied. The power dissipated by the rotator into the plasma and the breaking moment of forces acting on the dipole were calculated. Energetic processes in the system of the rotator were also calculated, whereby the plasma described in terms of equivalent resistances and inductances. The structure of the current system excited in the plasma, as well as the plasma particle average velocity field were investigated.

1 Introduction

There are a number of natural phenomena which can be considered as magnetic rotators in a plasma such as: pulsars [Goldreich and Julian, 1969; Ginzburg, 1986], planets, bipolar magnetic regions on the Sun, magnetic loops of antennas, plasma probes and inductors [Chugunov, 1973; Gubchenko, 1989], or artificial space satellites having their own magnetic moment. The results obtained below could be applied, in particular, to some cases of Jovian magnetospheric physics [Connerney, 1992], especially for the improvement of the models interpreting the Jovian radio emission [Connerney et al., 1981; Caudal and Connerney, 1989; Acuña and Ness, 1976a,b], by taking into account the structure of magnetic field lines in the planetary magnetosphere, because the analysis carried out here allows us to obtain a real structure of the field in the vicinity of a magnetic rotator. Besides,

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some results considered here could be useful for various tasks in the field of spacecraft technology. But every application of these results to a specific problem requires a special treatment to check the main idealizations and assumptions.

To simplify our analysis we assumed that the rotator is transparent for the plasma particles and that the plasma is isotropic and collisionless with the Maxwellian equilibrium initial distribution of particles. In general, this problem is non-linear. But if we consider the regions spaced far enough from the source, or weak sources, then the problem becomes linear and admits of an analytical treatment.

By inductive fields, produced by a magnetic dipole rotating with $\mathbf{\Omega}$, we mean quasi-stationary electromagnetic fields with a phase velocity $V_{ph} \sim [\mathbf{\Omega} \times \mathbf{r}]$ being much less than the thermal velocity of plasma particles $V_{T\alpha}$. This means that inductive fields are located deep inside the so-called thermal cylinder with the radius $r_{Tc} = V_{T\alpha}/\Omega$. In this region plasma appears as a conducting medium in which inductive currents are excited. The spatial behavior of fields inside the thermal cylinder is characterized by an anomalous skin scale

$$r_s = \left(\sum_{\alpha=i,e} \frac{\omega_{p\alpha}^2}{c^2} \frac{\Omega}{V_{T\alpha}} \sqrt{\frac{\pi}{2}} \right)^{-1/3} .$$

Effects, named above, are most pronounced and define the general structure of the fields and the energetics of the rotator at $r_s \ll r_{Tc}$, i.e., when considering rather slow rotators with $\Omega \ll \omega_{pe} V_{Te}/c$ [Rukhadze et al., 1988; Lifshitz and Pitaevsky, 1981].

2 General Solution

The magnetic dipole $\vec{\mu_0}$ distributed on the scale r_0, which rotates with ω and has an angle θ_0 with the rotation axis (see Figure 1) corresponds to the external source current

$$\mathbf{j}_0 = c \cdot \mathrm{rot}\, \boldsymbol{\mu}_\mathbf{0} \tag{1}$$

where

$$\boldsymbol{\mu}_\mathbf{0} = \mu_{G0}(r, r_0) \cdot \{sin(\Theta_0)[\mathbf{x}_0 cos(\Omega t) + \mathbf{y}_0 sin(\Omega t)] + \mathbf{z}_0 cos(\Theta_0)\}$$

and

$$\mu_{G0}(r, r_0) = \frac{\mu_0}{(2\pi r_0^2)^{3/2}} exp\left(-\frac{r^2}{2r_0^2} \right) .$$

Solving the set of Maxwell's equations with the external current given by Eq.(1) together with Vlasov's kinetic equation in the linear approximation by the Fourier-Laplace transform method, we obtain for the vector-potential field:

$$\mathbf{A}(\omega, \mathbf{k}) = \frac{4\pi \mathbf{j}_0(\omega, \mathbf{k})}{ck^2 D_T(\omega, \mathbf{k})} , \tag{2}$$

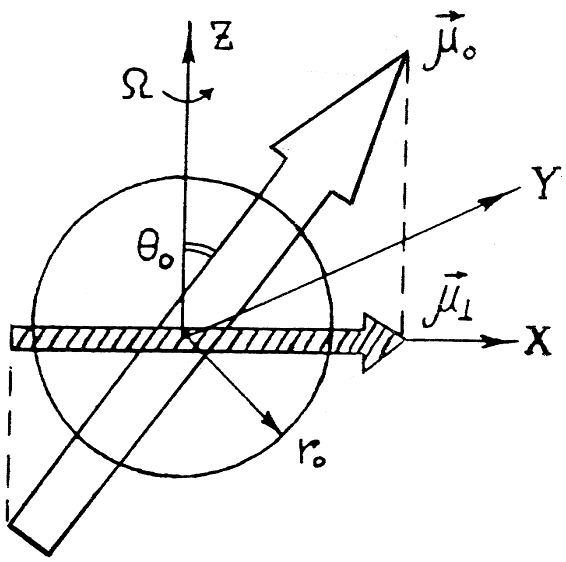

Figure 1: Rotating with Ω magnetic dipole μ_0, distributed on the scale r_0, and making the angle Θ_0 with the rotation axis.

where

$$D_T(\omega, \mathbf{k}) = 1 - \frac{\omega^2}{(ck)^2}\varepsilon_t(\omega, \mathbf{k}) \ . \tag{3}$$

Here $\varepsilon_t(\omega, \mathbf{k})$ is the transverse dielectric permittivity of a collisionless isotropic plasma. For the inductive electromagnetic field of interest the general expression Eq.(3) reduces to

$$D_T(\omega, k) = 1 - i\frac{k}{\mid k \mid}\frac{1}{k^3 r_s^3} \ , \tag{4}$$

whereby an asymptotical representation has been taken into account. At $kr_s \ll 1$ plasma behaves as an ideally conducting medium since the imaginary part of ε_t strongly increases. At $kr_s \gg 1$ plasma behaves similar to a nonconducting medium.

3 Analysis of the field $\mathbf{A}(\mathbf{r}, t)$ structure

After taking the inverse Fourier-Laplace transform in Eq.(2) we find that the vector-potential field at any fixed time can be represented as a superposition of two dipole harmonics:

$$\mathbf{A}(\mathbf{r}, t = t_0) = A_{d1}(r) \cdot \mathbf{\Psi}_{d1} + A_{d2}(r) \cdot \mathbf{\Psi}_{d2} \ , \tag{5}$$

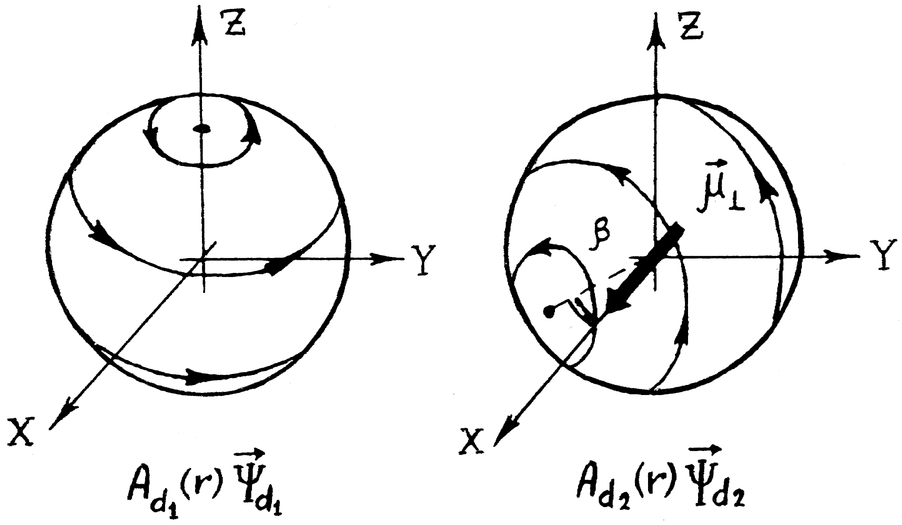

Figure 2: Instantaneous structure of vector-potential field can be represented as a superposition of two dipole harmonics. One of them describes the electromagnetic field of the co-axial component of the rotator in vacuum, and the other is caused by the rotation of the transverse component of the magnetic dipole in plasma.

where $\mathbf{\Psi}_{d1}$ and $\mathbf{\Psi}_{d2}$ are intrinsic for the dipole field angular dependencies $\mathbf{\Psi}_d = sin(\Theta)\cdot[-\mathbf{x}_0 sin(\varphi)+\mathbf{y}_0 cos(\varphi)]$ in coordinate systems having the z-axis coinciding with the symmetry axis of corresponding dipole harmonic (see Figure 2).

The first term in Eq.(5) is given by

$$A_{d1}(r)\cdot\mathbf{\Psi}_{d1} = \frac{2\mu_0 cos(\Theta_0)}{\pi}\cdot\left\{\frac{\pi}{2r^2}erf\left(\frac{r}{r_0\sqrt{2}}\right)-\sqrt{\frac{\pi}{2}}\frac{1}{rr_0}exp\left(-\frac{r^2}{2r_0^2}\right)\right\}\cdot\mathbf{\Psi}_{d1} \tag{6}$$

and describes the electromagnetic field of the co-axial component of the rotator $\boldsymbol{\mu}_{\|}$ in vacuum, and the second term

$$A_{d2}(r)\cdot\mathbf{\Psi}_{d2} = 2\mu_0 sin(\Theta_0)\frac{\mid a(r,Re_{ma})\mid}{\pi^2 r_s^2}\cdot\mathbf{\Psi}_{d2} \tag{7}$$

is due to the rotation of the transverse component $\boldsymbol{\mu}_\perp$ of the magnetic dipole. The axis of symmetry of the second dipole harmonic makes with the direction of the rotator's transverse component the angle

$$\beta(r) = -arctg\left(\frac{Im(a(r,Re_{ma}))}{Re(a(r,Re_{ma}))}\right)\ . \tag{8}$$

The radial dependence of the fields is determined by an integral characteristic function

$$a(r,Re_{ma}) = \int_0^\infty \frac{\xi^4 exp(-\xi^2 Re_{ma}^2/2)}{\xi^3 - i} j_1(\xi r/r_s)d\xi\ , \tag{9}$$

where $j_1(x)$ is spherical Bessel function, $\xi = kr_s$ and $Re_{ma} = r_0/r_s$ is anomalous collisionless magnetic Reynolds number, characterizing the role of the effects of magnetic viscosity and conductivity in a collisionless plasma.

The integral $a(r, Re_{ma})$ can be estimated in some regions of interest:

1. far region, where $r \gg r_s, r_0$
2. near source region $r \sim 0$;
   (a) $r \ll r_s \ll r_0, \quad (Re_{ma} \to \infty)$
   (b) $r \ll r_0 \ll r_s, \quad (Re_{ma} \to 0)$
   (c) $r_0 \ll r \ll r_s, \quad (Re_{ma} \to 0)$

When $r \gg r_0, r_s$ the source can be considered as a point source. Using the residue theory we find:

$$a(r, Re_{ma}) \approx \frac{\pi}{3} \cdot \left(\frac{r_s}{r}\right)^2 exp\left(i\frac{\sqrt{3}}{2}\frac{r}{r_s} - \frac{r}{2r_s}\right) + \frac{\pi}{6} exp\left(-\frac{r}{r_s}\right) \cdot \frac{r_s}{r}\left[\frac{r_s}{r} + 1\right] +$$

$$\frac{\pi}{3}\frac{r_s}{r} exp\left(i\frac{\sqrt{3}}{2}\frac{r}{r_s} - i\frac{\pi}{3} - \frac{r}{2r_s}\right) - i\sum_{n=0}^{\infty}\left(\frac{r_s}{r}\right)^{5+6n}[(2+6n)! + (3+6n)!]\ . \qquad (10)$$

Thus, the field $\mathbf{A}_{d2}$ of the rotating dipole transverse component, which decays as $1/r^2$ in a non-conducting medium, begins to skin into the plasma on a characteristic scale r_s. The field decay law changes so that at $r \to \infty$, we have $A_{d2}(r) \propto 1/r^5$, and the angle β tends to $-\pi/2$.

In the near source region, assuming $r/r_s \ll 1$ with $Re_{ma} \gg 1$ we obtain

$$a(r, Re_{ma}) \approx \frac{r}{r_s} \cdot \left\{\frac{1}{6}\left(\frac{Re_{ma}}{2}\right)^{-9/2}\Gamma(9/2) + \frac{i}{6}\left(\frac{Re_{ma}}{2}\right)^{-3}\Gamma(3) + O(Re_{ma}^{-6})\right\} + O(r^3/r_s^3)\ . \qquad (11)$$

In the case $r \ll r_0$ with $Re_{ma} \ll 1$ the function $a(r, Re_{ma})$ is represented by

$$a(r, Re_{ma}) \approx \frac{r}{r_s}\frac{1}{3}\left\{\sqrt{\frac{\pi}{2}}Re_{ma}^{-3/2} - (\frac{\pi}{6} + O(Re_{ma})) - \frac{i}{2}(ln\left(\frac{Re_{ma}}{2}\right) + O(Re_{ma}))\right\} - O(r^3/r_s^6)\ . \qquad (12)$$

In the region $r_0 \ll r \ll r_s$, far enough from the source, which can be assumed as a point one, the function $a(r, Re_{ma})$ has the form

$$a(r, Re_{ma}) \approx \frac{\pi}{3}\left(\frac{r_s}{r}\right)^2 exp\left(i\frac{\sqrt{3}}{2}\frac{r}{r_s} - \frac{r}{2r_s}\right) + \frac{\pi}{6}exp\left(-\frac{r}{r_s}\right)\left(\frac{r_s}{r}\right)\cdot\left[\frac{r_s}{r}+1\right]+$$

$$\frac{\pi}{3}\frac{r_s}{r}exp\left(i\frac{\sqrt{3}}{2}\frac{r}{r_s} - i\frac{\pi}{3} - \frac{r}{2r_s}\right) + i\left(-\frac{\pi\sqrt{3}}{12} - O(r/r_s)\right) \ . \quad (13)$$

From the Equations (12) and (13) it follows, in view of Eq.(7), that at a distance $r \ll r_s$ from the source the field of the dipole differs little from the vacuum field; the field corrections are due to the weak influence of a current system excited in plasma.

4 Structure of Current System and Particle Average Velocity Field

Using the relations $\mathbf{J} = \frac{c}{4\pi}\nabla^2(\mathbf{A}_p)$, where $\mathbf{A}_p = \mathbf{A} - \mathbf{A}_0$, $\mathbf{A}_0$ is the source vector-potential in vacuum, and $< \mathbf{V}_\alpha >= \int \mathbf{V} f_\alpha(\mathbf{V}, \mathbf{r}, t) d\mathbf{V}$ the excited current system and plasma particles average velocity field at any fixed time may be presented by one dipole harmonic

$$\mathbf{j}_p = \frac{c\mu_0 sin(\Theta_0)}{2\pi^2 r_s^4} \mid f(r, Re_{ma}) \mid \cdot \mathbf{\Psi}_d \quad (14)$$

$$< \mathbf{V}_\alpha >= \frac{q_\alpha \mu_0 sin(\Theta_0)}{m_\alpha c V_{T\alpha}} \frac{\Omega}{r_s}\sqrt{\frac{2}{\pi}} \mid f(r, Re_{ma}) \mid \cdot \mathbf{\Psi}_d \ . \quad (15)$$

They appear as a complex of spherical vortexes enclosing each other, which move simultaneously with the rotator, and their axes make with the direction of the transverse component of the rotator $\boldsymbol{\mu}_\perp$ the angle $\gamma(r) = arctg[Im(f(r, Re_{ma}))/Re(f(r, Re_{ma}))] - \pi/2$ to the $\mu_\perp$ (see Figure 3). The function

$$f(r, Re_{ma}) = \int_0^\infty \frac{\xi^3 exp(-\xi^2 Re_{ma}^2/2)}{\xi^3 + i} j_1(\xi r/r_s) d\xi \quad (16)$$

characterizes the spatial dependence of these fields. Using the asymptotic representation of $f(r, Re_{ma})$ at great distances it can be shown that far from the source $\gamma(r) \to -\pi/2$.

The total magnetic moment excited by the rotator currents

$$\mathbf{m} = \frac{1}{2c}\int [\mathbf{r} \times \mathbf{j}_p] d\mathbf{r} \quad (17)$$

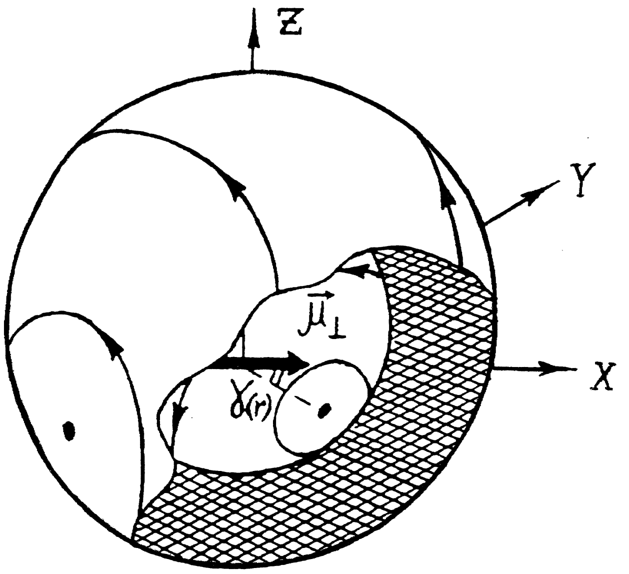

Figure 3: Structure of current system and particle average velocity field appears as a complex of enclosing each other spherical vortexes, which move simultaneously with the rotator.

is equal in module and opposed to the rotator's transverse component $\boldsymbol{\mu}_\perp$ at any fixed time. This fact means that transverse component of the rotator is shielded by the plasma currents.

The value of total mechanical momentum of plasma particles $\mathbf{L}_\alpha = m_\alpha n_\alpha \int [\mathbf{r} \times \mathbf{V}_\alpha] f_\alpha(\mathbf{V}, \mathbf{r}, t) d\mathbf{V} d\mathbf{r}$ at any fixed time is given by the following formula:

$$| \mathbf{L}_\alpha |= -\frac{8\pi^2}{\sqrt{2\pi}} \frac{m_\alpha n_\alpha q_\alpha V_\alpha \mu_0 \Omega r_s^3}{c\kappa T} sin(\Theta_0) \ . \tag{18}$$

The mechanical momentum vector is collinear to the transverse component of the rotator and moves simultaneously with it. From Eq.(17) and Eq.(18) a natural relationship follows:

$$\mathbf{L}/\mathbf{m} = 8\pi c \frac{\sum_\alpha n_\alpha q_\alpha}{\sum_\alpha \omega_{p\alpha}^2} \ . \tag{19}$$

5 Structure of the Magnetic Field

In the case of a plane-rotating magnetic dipole ($\Theta_0 = \pi/2$) at any fixed time the equation for the field line in the equatorial plane of the rotator is

$$rB_r d\varphi = B_\varphi dr \ , \tag{20}$$

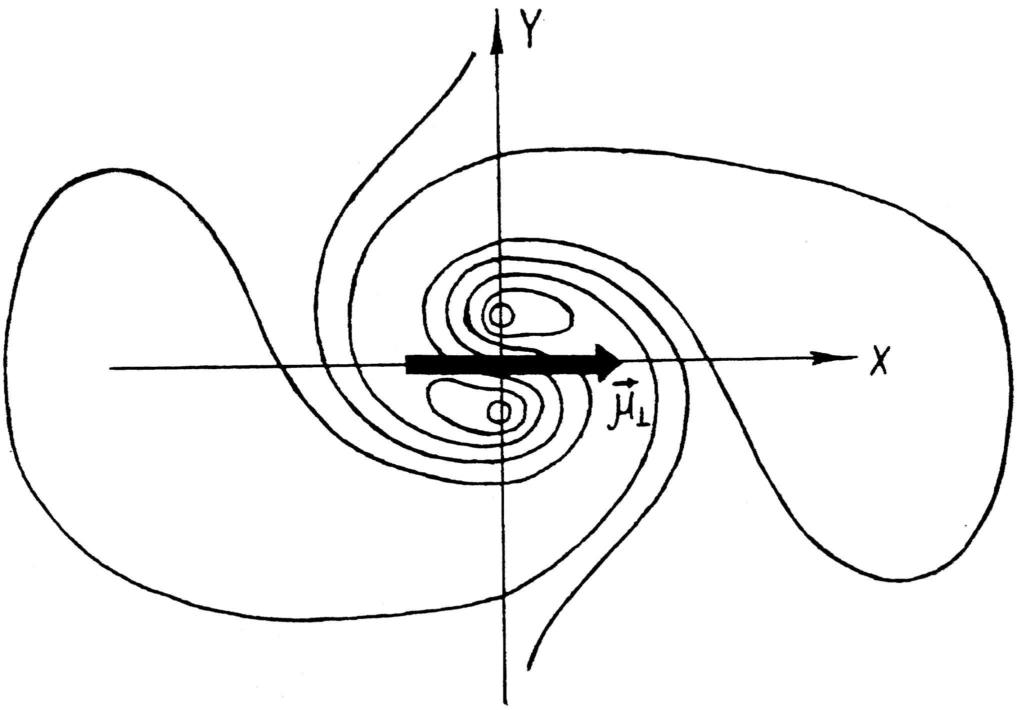

Figure 4: Instantaneous picture of magnetic field lines in the equatorial plane of the rotator.

where

$$B_r = \frac{2}{r}\frac{2\mu_0}{\pi r_s^2}\frac{\partial}{\partial\varphi}Im(a(r, Re_{ma})e^{i\varphi}) \ , \tag{21}$$

$$B_\varphi = -\frac{2\mu_0}{\pi r_s^2}\frac{\partial}{\partial r}Im(a(r, Re_{ma})e^{i\varphi}) - \frac{1}{r}\frac{2mu_0}{\pi r_s^2}Im(a(r, Re_{ma})e^{i\varphi}) \ . \tag{22}$$

The solution of Equation (20) gives us a magnetic field line equation in the equatorial plane of the rotator

$$\varphi = -arctg\left(\frac{Im(a(r, Re_{ma}))}{Re(a(r, Re_{ma}))}\right) \pm arcsin\left(\frac{const}{\sqrt{ra(r, Re_{ma})}}\right) \ . \tag{23}$$

The instantaneous picture of magnetic field lines in the equatorial plane of the rotator is shown in Figure 4.

In the near source region the rotator's field is similar to the one in vacuum. Then the influence of currents excited in plasma becomes considerable, and the field line picture changes. In the limiting case $r \gg r_s, r_0$ the field line equation, after the asymptotical representation of the function $a(r, Re_{ma})$ has been taken into account, is
$r = const \cdot \sqrt{\mid cos\varphi \mid}$.

6 Energetics of the Rotator and its electrical characteristics

Determining the power dissipated by the rotator $P = -\int \mathbf{j}_0 \cdot \mathbf{E}d\mathbf{r}$, we can introduce the loss resistance $R_\Sigma = P/I_0$, where $I_0 = \int \mathbf{j}_0 d\mathbf{s} = \dfrac{c\mu_0}{2\pi r_0^2}$ is the source current. In other words, we consider the plasma as a load for the generator, producing the rotating magnetic moment [Chugunov, 1973; Gubchenko, 1989]. The dependence of this loss resistance on the source dimensions and plasma parameters is a function of collisionless Reynolds number $Re_{ma} = r_0/r_s$:

$$R_\Sigma = \frac{8\pi}{3}\frac{\Omega r_0}{c^2} sin^2\Theta_0 Re_{ma}^3 \int_0^\infty \frac{e^{-y}y^2}{y^3 + Re_{ma}^6} dy \ . \tag{24}$$

In the limiting cases we have

$$R_\Sigma(Re_{ma} \to 0) \approx -\frac{8\pi}{3}\frac{\Omega r_0}{c^2} sin^2\Theta_0 2Re_{ma}^3 ln(Re_{ma}) \ , \tag{25}$$

$$R_\Sigma(Re_{ma} \to \infty) \approx \frac{8\pi}{3}\frac{\Omega r_0}{c^2} sin^2\Theta_0 2Re_{ma}^{-3} \ . \tag{26}$$

The behavior of the loss resistance as function of the collisionless Reynolds number is shown in Figure 5. This resistance describes the inductive field generation efficiency and is not associated with the radiation. It characterizes the energy losses due to the resonant plasma particles acceleration [Rukhadze et al., 1988]. At $Re_{ma} \to 0$ and $Re_{ma} \to \infty$ the loss resistance tends to zero because there are no losses in a non-conducting and ideally conducting medium.

The breaking moment of forces, acting the rotator is equal $\mathbf{N} = -\mathbf{z}_0 P/\Omega$.

Considering the plasma as an equivalent circuit with an inductance L_p which interacts with the mutual induction coefficient M with the circuit L_v, producing the magnetic moment of the source one can find the magnetic energy of such a system

$$W = \frac{1}{8\pi}\int B^2 d\mathbf{r} = \frac{L_v I_0^2}{2c^2} + \frac{L_p I_p^2}{2c^2} + \frac{M I_0 I_p}{2c^2} \ , \tag{27}$$

and express these inductive characteristics:

$$L_p = \frac{8\pi}{3} r_0 sin^2\Theta_0 Re_{ma}^6 \int_0^\infty \frac{e^{-y}\sqrt{y}}{y^3 + Re_{ma}^6} dy \ , \tag{28}$$

$$L_v = \frac{4\pi\sqrt{\pi}}{3} r_0 sin^2\Theta_0, \quad M = -2L_p \ . \tag{29}$$

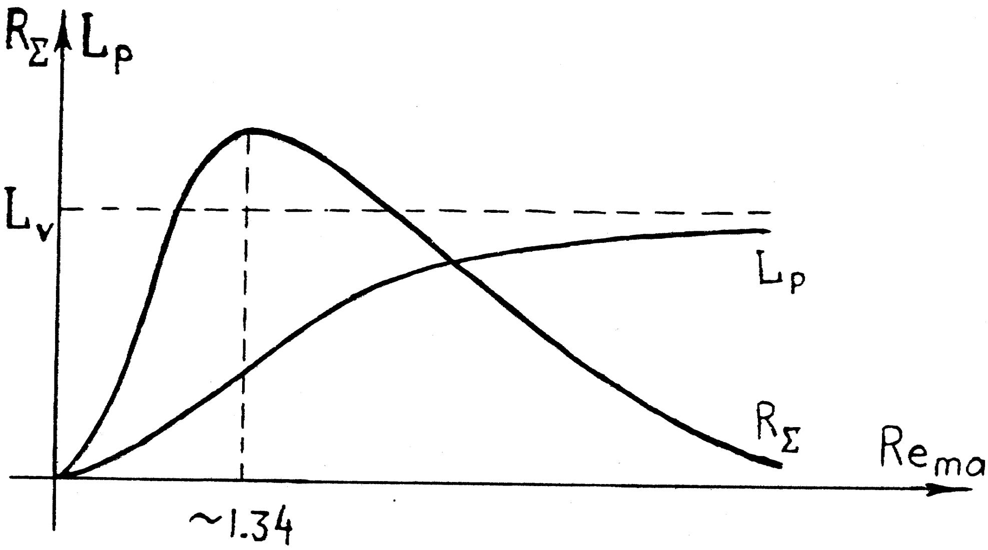

Figure 5: Behavior of the loss resistance R_Σ, and plasma inductance L_p as function of the collisionless Reynolds number.

The behavior of the plasma inductance L_p as a function of the collisionless Reynolds number is shown in Figure 5. Asymptotical representations of inductive characteristics in the limiting cases are as follows:

$$L_p(Re_{ma} \to 0) \approx \frac{8\pi}{3} r_0 sin^2\Theta_0 \frac{\pi}{3} Re_{ma}^3 \tag{30}$$

$$L_p(Re_{ma} \to \infty) \approx \frac{8\pi}{3} r_0 sin^2\Theta_0 \left(\frac{\sqrt{\pi}}{2} - \frac{7!!\sqrt{\pi}}{16} Re_{ma}^{-6} \right) \approx L_v \ . \tag{31}$$

From Eq. (30) and Eq. (31) it follows that in a non-conducting medium ($Re_{ma} \to 0$) its inductances are equal to zero, and for an ideally conducting medium ($Re_{ma} \to \infty$) $L_p \to L_v$. This is because the currents excited in an ideally conducting medium are located very close to the circuit of the source and practically repeat its shape.

7 Conclusion

In conclusion we shall consider the applicability of the results to real situations.

1. When investigating a magnetic rotator one has usually to solve the problem of the interaction of a rotating magnetized body with plasma [Goldreich and Julian, 1969; Lifshitz and Pitaevsky, 1981]. In a laboratory reference system, together with a magnetic and a vortical electric field there is an electrostatic. Owing to the isotropy of the plasma in this linear problem the potential electric field does not influence the excitation of the

vortical electromagnetic fields by the external current $\mathbf{j} = c \cdot \mathrm{rot}\,\mu$ and therefore was not investigated.

2. In the case of a magnetized plasma the results obtained above are valid if $r_s \ll r_{H\alpha}$. This condition corresponds to the inequality $\dfrac{\beta\Omega}{\omega_{H\alpha}} \gg 1$, where β is the plasma parameter. If this condition is violated, one should take into account the tensor character of the plasma permittivity.

3. In the case of non-stationarity and non-uniformity of the plasma and the source parameters the steady-state solutions are valid for $\Delta t \gg \tau$ and $R \gg r_s$, where Δt, R are the characteristic scales of variation of the parameters, and $\tau = \dfrac{1}{\mid \omega(1/r^*) \mid}$ is a characteristic time of relaxation to a stead state with the field scale r^*. In the case of large Re_{ma}, assuming $r^* \sim r_0$ we have $\tau = (Re_{ma})^3/\Omega$; at small Re_{ma}, putting $r^* \sim r_s$, we have $\tau = 1/\Omega$.

4. The collisions effects can be neglected if the particle mean free path is much greater than r_s. This means that the rotator frequency should satisfy the inequality $\Omega \gg \dfrac{\nu_\alpha^3 c^2}{(\omega_{p\alpha} V_\alpha)^2}$.

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