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ANALYSIS OF THE MULTIPLICATIVE SCHWARZ METHOD FOR MATRICES WITH A SPECIAL BLOCK STRUCTURE*

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Abstract. We analyze the convergence of the (algebraic) multiplicative Schwarz method applied to linear algebraic systems with matrices having a special block structure that arises, for example, when a (partial) differential equation is posed and discretized on a two-dimensional domain that consists of two subdomains with an overlap. This is a basic situation in the context of domain decomposition methods. Our analysis is based on the algebraic structure of the Schwarz iteration matrices, and we derive error bounds that are based on the block diagonal dominance of the given system matrix. Our analysis does not assume that the system matrix is symmetric (positive definite), or has the M- or H-matrix property. Our approach is motivated by, and significantly generalizes, an analysis for a special one-dimensional model problem of Echeverría et al. given in [Electron. Trans. Numer. Anal., 48 (2018), pp. 40–62].

Key words. multiplicative Schwarz method, iterative methods, convergence analysis, singularly perturbed problems, Shishkin mesh discretization, block diagonal dominance

AMS subject classifications. 15A60, 65F10, 65F35

1. Introduction.

The (algebraic) multiplicative Schwarz method, sometimes also called the Schwarz alternating method, is a stationary iterative method for solving large and sparse linear algebraic systems

$$\mathcal{A}x = b. \tag{1.1}$$

In each step of the method, the current iterate is multiplied by an iteration matrix that is the product of several factors, where each factor corresponds to an inversion of only a restricted part of the matrix $\mathcal{A}$. In the context of the numerical solution of discretized differential equations, the restrictions of the matrix correspond to different parts of the computational domain. This motivates the name "local solve" given to each factor, which is also used in a purely algebraic setting.

The convergence theory for the multiplicative Schwarz method is well established for important matrix classes including symmetric positive definite matrices and nonsingular M-matrices [1], symmetric indefinite [6, 7] and semidefinite [14] matrices, and H-matrices [2]. The derivation of convergence results for these matrix classes is usually based on splittings of $\mathcal{A}$. No systematic convergence theory exists however for general nonsymmetric matrices.

An important source of nonsymmetric linear algebraic systems is the discretization of singularly perturbed convection-diffusion problems. In a recent paper we have studied the convergence of the multiplicative Schwarz method for a one-dimensional model problem in this context [4]. The system matrices in this problem are usually nonsymmetric, nonnormal, ill-conditioned, and in particular not in one of the classes considered in [1, 2, 6, 7]. Moreover, rather than using classical tools from the theory of matrix splittings, our convergence analysis in [4] is based on the off-diagonal decay of the matrix inverses, which in turn is implied by diagonal dominance. From a broader point of view, our results show why a convergence theory for the multiplicative Schwarz method for "general" matrices will most likely remain elusive:

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even in the simple model problem considered in [4], the convergence of the method strongly depends on the problem parameters and on the chosen discretization, and while the method rapidly converges in some cases, it even diverges in others.

Ideally, the study of well chosen model problems involves a "generalizability aspect" in the sense that the obtained results shall give insight into more general problems; see for example the interesting discussion about the nature of model problems in [11]. This is also true for the approach in [4], since it motivated our analysis in this paper, which may be considered a significant generalization of the previous work. Here we analyze the convergence of the multiplicative Schwarz method for linear algebraic systems with matrices of the form

$$\mathcal{A} = \begin{bmatrix} \widehat{A}_H & e_m \otimes B_H & 0 \\ e_m^T \otimes C & A & e_1^T \otimes B \\ 0 & e_1 \otimes C_h & \widehat{A}_h \end{bmatrix} \in \mathbb{R}^{N(2m+1)\times N(2m+1)}, \tag{1.2}$$

with $\widehat{A}_H, \widehat{A}_h \in \mathbb{R}^{Nm\times Nm}$, $A, B, C, B_H, C_h \in \mathbb{R}^{N\times N}$, and the canonical basis vectors $e_1, e_m \in \mathbb{R}^m$. We will usually think of $\widehat{A}_H, \widehat{A}_h \in \mathbb{R}^{Nm\times Nm}$ as matrices consisting of m blocks of size $N \times N$. After deriving general expressions for the norms of the multiplicative Schwarz iteration matrices for systems of the form (1.1)–(1.2), we derive actual error bounds only for the case when the blocks $\widehat{A}_H$ and $\widehat{A}_h$ of $\mathcal{A}$ are block tridiagonal.

Such matrices arise naturally, for example, when a differential equation is posed inside a two-dimensional domain Ω that is divided into two subdomains Ω_1 and Ω_2 with an overlap, and discretized using finite-differences on a rectangular grid. In this context the first $m+1$ block rows in the matrix $\mathcal{A}$ correspond to the unknowns in the domain Ω_1, the last $m+1$ block rows correspond to the unknowns in the domain Ω_2, and the middle block row corresponds to the unknowns in the overlap. The underlying assumption here is that in each of the two subdomains we have the same number of unknowns. This assumption is made for simplicity of our exposition. Extensions to other block sizes are possible, but would require even more technicalities; see our discussion in Section 6. We point out that the model problems studied in [4] are of the form (1.1)–(1.2) with $N = 1$. While (usual) diagonal dominance of tridiagonal matrices is one of the main tools in [4], the derivation of error bounds here relies on recent results on block diagonal dominance of block tridiagonal matrices from [3].

This paper is organized as follows. In Section 2 we state the multiplicative Schwarz method for linear algebraic systems of the form (1.1)–(1.2), and in Section 3 we study the algebraic structure and the norm of its iteration matrices. In Section 4 we derive error bounds for the method when the matrix $\mathcal{A}$ is block tridiagonal and block diagonally dominant. We apply these error bounds in Section 5 to a two-dimensional discretized convection-diffusion model problem. Finally, in Section 6 we summarize the main results of the paper and briefly discuss possible generalizations and alternative applications of our approach.

2. The multiplicative Schwarz method.

The multiplicative Schwarz method for solving linear algebraic systems of the form (1.1)–(1.2) can naturally be based on two local solves using the top and the bottom $N(m+1)\times N(m+1)$ blocks of $\mathcal{A}$, respectively. More precisely, the restriction operators of the method are

$$R_1 \equiv \begin{bmatrix} I_{N(m+1)} & 0 \end{bmatrix} \quad \text{and} \quad R_2 \equiv \begin{bmatrix} 0 & I_{N(m+1)} \end{bmatrix},$$

which have both size $N(m+1)\times N(2m+1)$. The corresponding restrictions of $\mathcal{A}$ are

$$A_1 \equiv R_1 \mathcal{A} R_1^T = \begin{bmatrix} \widehat{A}_H & e_m \otimes B_H \\ e_m^T \otimes C & A \end{bmatrix} \text{ and } A_2 \equiv R_2 \mathcal{A} R_2^T = \begin{bmatrix} A & e_1^T \otimes B \\ e_1 \otimes C_h & \widehat{A}_h \end{bmatrix},$$

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which have both size $N(m+1) \times N(m+1)$. We now define the two projections

$$P_i \equiv R_i^T A_i^{-1} R_i \mathcal{A} \in \mathbb{R}^{N(2m+1)\times N(2m+1)}, \quad i = 1, 2. \tag{2.1}$$

Then their complementary projections

$$Q_i \equiv I - P_i \in \mathbb{R}^{N(2m+1)\times N(2m+1)}, \quad i = 1, 2,$$

yield the multiplicative Schwarz iteration matrices

$$T_{12} \equiv Q_2 Q_1 \quad \text{and} \quad T_{21} \equiv Q_1 Q_2.$$

If, in the context of discretized differential equations, the top and bottom blocks of $\mathcal{A}$ correspond to the unknowns in the domains Ω_1 and Ω_2, respectively, then a multiplication with the matrix T_{ij} corresponds to correcting the iterate by a contribution obtained from a local solve of the residual equation first on Ω_i, and then on Ω_j.

We now fix either T_{12} or T_{21}, and choose an initial vector $x^{(0)} \in \mathbb{R}^{N(2m+1)}$. Then the (algebraic, one-level) multiplicative Schwarz method is defined by

$$x^{(k+1)} = T_{ij} x^{(k)} + v, \quad k = 0, 1, 2, \ldots . \tag{2.2}$$

The vector $v \in \mathbb{R}^{N(2m+1)}$ is defined to make the method consistent. The consistency condition is given by $x = T_{ij}x + v$, and this yields

$$v = (I - T_{ij})x = (P_1 + P_2 - P_j P_i)x,$$

which is (easily) computable since

$$P_i x = R_i^T A_i^{-1} R_i \mathcal{A} x = R_i^T A_i^{-1} R_i b, \quad i = 1, 2.$$

The error of the multiplicative Schwarz iteration (2.2) is given by

$$e^{(k+1)} = x - x^{(k+1)} = (T_{ij}x + v) - (T_{ij}x^{(k)} + v) = T_{ij} e^{(k)}, \quad k = 0, 1, 2, \ldots,$$

and hence $e^{(k+1)} = T_{ij}^{k+1} e^{(0)}$ by induction. For any consistent norm $\|\cdot\|$, we therefore have the error bound

$$\|e^{(k+1)}\| \le \|T_{ij}^{k+1}\| \, \|e^{(0)}\|. \tag{2.3}$$

In the following we will use the structure of the iteration matrices T_{ij} to derive bounds on the norms $\|T_{ij}^{k+1}\|$. This will lead to stronger convergence results than the standard approach in the convergence analysis of stationary iterative methods, which uses submultiplicativity, i.e., $\|T_{ij}^{k+1}\| \le \|T_{ij}\|^{k+1}$, and then bounds $\|T_{ij}\|$. As mentioned above, our approach is motivated by [4], but since we consider matrices with block rather than scalar entries, neither the analysis nor the results from [4] are directly applicable here.

3. Structure and norms of the iteration matrices. Let us have a closer look at the structure of the matrices T_{ij}. A direct computation based on (2.1) shows that

$$P_1 = \begin{bmatrix} I_{N(m+1)} \\ 0 \end{bmatrix} A_1^{-1} \left[\, A_1 \mid e_{m+1} \otimes B \mid 0 \,\right] = \begin{bmatrix} I_{N(m+1)} & A_1^{-1}(e_{m+1} \otimes B) & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$P_2 = \begin{bmatrix} 0 \\ I_{N(m+1)} \end{bmatrix} A_2^{-1} \left[\, 0 \mid e_1 \otimes C \mid A_2 \,\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_2^{-1}(e_1 \otimes C) & I_{N(m+1)} \end{bmatrix},$$

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where $e_1, e_{m+1} \in \mathbb{R}^{m+1}$. We see that both P_1 and P_2 have exactly $N(m+1)$ linearly independent columns, and hence

$$\operatorname{rank}(P_1) = \operatorname{rank}(P_2) = N(m+1).$$

Moreover, the complementary projections are

$$Q_1 = \begin{bmatrix} 0 & -A_1^{-1}(e_{m+1}\otimes B) & 0 \\ 0 & I_N & 0 \\ 0 & 0 & I_{N(m-1)} \end{bmatrix}, \; Q_2 = \begin{bmatrix} I_{N(m-1)} & 0 & 0 \\ 0 & I_N & 0 \\ 0 & -A_2^{-1}(e_1\otimes C) & 0 \end{bmatrix},$$

and we have

$$\operatorname{rank}(Q_1) = \operatorname{rank}(Q_2) = Nm.$$

In order to simplify the notation we write

$$\begin{bmatrix} P^{(1)} \\ \Pi^{(1)} \end{bmatrix} \equiv A_1^{-1}(e_{m+1}\otimes B) \quad \text{and} \quad \begin{bmatrix} \Pi^{(2)} \\ P^{(2)} \end{bmatrix} \equiv A_2^{-1}(e_1\otimes C), \tag{3.1}$$

where $\Pi^{(i)} \in \mathbb{R}^{N\times N}$, and

$$P^{(i)} = \left[\left(P_1^{(i)}\right)^T, \ldots, (P_m^{(i)})^T \right]^T \in \mathbb{R}^{Nm\times N}, \quad \text{with} \quad P_j^{(i)} \in \mathbb{R}^{N\times N}, \; j = 1, \ldots, m.$$

Then

$$Q_1 = \begin{bmatrix} 0 & -P^{(1)} & 0 \\ 0 & -\Pi^{(1)} & 0 \\ 0 & I_N & 0 \\ 0 & 0 & I_{N(m-1)} \end{bmatrix} \quad \text{and} \quad Q_2 = \begin{bmatrix} I_{N(m-1)} & 0 & 0 \\ 0 & I_N & 0 \\ 0 & -\Pi^{(2)} & 0 \\ 0 & -P^{(2)} & 0 \end{bmatrix},$$

so that

$$\begin{aligned} T_{12} = Q_2 Q_1 &= \begin{bmatrix} 0 & -P^{(1)} & 0 \\ 0 & \Pi^{(2)} P_m^{(1)} & 0 \\ 0 & P^{(2)} P_m^{(1)} & 0 \end{bmatrix} \\ &= \begin{bmatrix} -P^{(1)} \\ \Pi^{(2)} P_m^{(1)} \\ P^{(2)} P_m^{(1)} \end{bmatrix} \left[\, 0_{N(m+1)} \;\middle|\; I_N \;\middle|\; 0_{N(m-1)} \,\right] \equiv V_1 (e_{m+2}^T \otimes I_N), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} T_{21} = Q_1 Q_2 &= \begin{bmatrix} 0 & P^{(1)} P_1^{(2)} & 0 \\ 0 & \Pi^{(1)} P_1^{(2)} & 0 \\ 0 & -P^{(2)} & 0 \end{bmatrix} \\ &= \begin{bmatrix} P^{(1)} P_1^{(2)} \\ \Pi^{(1)} P_1^{(2)} \\ -P^{(2)} \end{bmatrix} \left[\, 0_{N(m-1)} \;\middle|\; I_N \;\middle|\; 0_{N(m+1)} \,\right] \equiv V_2 (e_m^T \otimes I_N), \end{aligned} \tag{3.3}$$

where $e_m, e_{m+2} \in \mathbb{R}^{2m+1}$ and $V_1, V_2 \in \mathbb{R}^{N(2m+1)\times N}$.

Using these representations of the matrices T_{ij}, we can obtain the following generalization of [4, Proposition 4.1 and Corollary 4.2].

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LEMMA 3.1. *In the notation established above we have* $\mathrm{rank}(T_{ij}) \le N$ *and*

$$T_{12}^{k+1} = V_1 \left(P_1^{(2)} P_m^{(1)}\right)^k (e_{m+2}^T \otimes I_N), \quad T_{21}^{k+1} = V_2 \left(P_m^{(1)} P_1^{(2)}\right)^k (e_m^T \otimes I_N), \tag{3.4}$$

for all $k \ge 0$.

Proof. We only consider the matrix T_{12}; the proof for T_{21} is analogous. The result about the rank is obvious from (3.2). We denote $E_{m+2} \equiv e_{m+2}^T \otimes I_N$, then $T_{12} = V_1 E_{m+2}$, and it is easy to see that

$$T_{12}^{k+1} = V_1 \left(E_{m+2} V_1\right)^k E_{m+2}, \quad \text{for all } k \ge 0.$$

Now

$$E_{m+2} V_1 = (e_{m+2}^T \otimes I_N) \begin{bmatrix} -P^{(1)} \\ \Pi^{(2)} P_m^{(1)} \\ P_1^{(2)} P_m^{(1)} \\ P_{2:m}^{(2)} P_m^{(1)} \end{bmatrix} = P_1^{(2)} P_m^{(1)},$$

which shows the first equality in (3.4). □

The next result generalizes [4, Lemma 4.3] and gives expressions for some of the block entries of the matrices T_{ij}, which will be essential in our derivations of error bounds in the following section.

LEMMA 3.2. *Suppose that the matrices* $\widehat{A}_H, \widehat{A}_h \in \mathbb{R}^{Nm \times Nm}$ *in* (1.2) *are nonsingular, and denote* $\widehat{A}_H^{-1} = [Z_{ij}^{(H)}]$ *and* $\widehat{A}_h^{-1} = [Z_{ij}^{(h)}]$ *with* $Z_{ij}^{(H)}, Z_{ij}^{(h)} \in \mathbb{R}^{N \times N}$. *Then, in the notation established above,*

$$\begin{bmatrix} P^{(1)} \\ \Pi^{(1)} \end{bmatrix} = \begin{bmatrix} -Z_{1:m,m}^{(H)} B_H \\ I_N \end{bmatrix} \Pi^{(1)}, \quad \Pi^{(1)} = \left(A - C Z_{mm}^{(H)} B_H\right)^{-1} B,$$

and

$$\begin{bmatrix} \Pi^{(2)} \\ P^{(2)} \end{bmatrix} = \begin{bmatrix} I_N \\ -Z_{1:m,1}^{(h)} C_h \end{bmatrix} \Pi^{(2)}, \quad \Pi^{(2)} = \left(A - B Z_{11}^{(h)} C_h\right)^{-1} C.$$

Proof. From (3.1) we know that $P^{(1)}$, $P^{(2)}$, $\Pi^{(1)}$, and $\Pi^{(2)}$ solve the linear algebraic systems

$$\begin{bmatrix} \widehat{A}_H & e_m \otimes B_H \\ e_m^T \otimes C & A \end{bmatrix} \begin{bmatrix} P^{(1)} \\ \Pi^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ B \end{bmatrix}, \qquad \begin{bmatrix} A & e_1^T \otimes B \\ e_1 \otimes C_h & \widehat{A}_h \end{bmatrix} \begin{bmatrix} \Pi^{(2)} \\ P^{(2)} \end{bmatrix} = \begin{bmatrix} C \\ 0 \end{bmatrix}.$$

Hence the expressions for $P^{(1)}$, $P^{(2)}$, $\Pi^{(1)}$, and $\Pi^{(2)}$ can be obtained using Schur complements; see, e.g., [8, § 0.7.3]. □

In order to bound norms of powers of the iteration matrices T_{ij} we have to decide first which matrix norm should be taken. In the following we use a general induced matrix norm $\|\cdot\|$ which can be considered for square as well as for rectangular matrices. Note that an induced matrix norm for square matrices is submultiplicative and satisfies $\|I\| = 1$.

LEMMA 3.3. *In the notation established above, for any induced matrix norm we have*

$$\|T_{ij}^{k+1}\| \le \rho_{ij}^k \|T_{ij}\|, \quad \textit{for all } k \ge 0, \tag{3.5}$$

where

$$\rho_{12} = \|Z_{11}^{(h)} C_h \Pi^{(2)} Z_{mm}^{(H)} B_H \Pi^{(1)}\| \quad \textit{and} \quad \rho_{21} = \|Z_{mm}^{(H)} B_H \Pi^{(1)} Z_{11}^{(h)} C_h \Pi^{(2)}\|. \tag{3.6}$$

Proof. We only consider the matrix T_{12}; the proof for T_{21} is analogous. For any vector $y = [y_1^T, \dots, y_{2m+1}^T]^T \in \mathbb{R}^{N(2m+1)}$ with block components $y_1, \dots, y_{2m+1} \in \mathbb{R}^N$ we have

$$T_{12}^{k+1} y = V_1 \left(P_1^{(2)} P_m^{(1)}\right)^k (e_{m+2}^T \otimes I_N) y = V_1 \left(P_1^{(2)} P_m^{(1)}\right)^k y_{m+2},$$

and therefore

$$\|T_{12}^{k+1}\| = \max_{\substack{y \in \mathbb{R}^{N(2m+1)} \\ \|y\|=1}} \|T_{12}^{k+1} y\| = \max_{\substack{y \in \mathbb{R}^{N} \\ \|y\|=1}} \|V_1 (P_1^{(2)} P_m^{(1)})^k y\| \le \rho_{12}^k \|V_1\|,$$

where $\rho_{12} \equiv \|P_1^{(2)} P_m^{(1)}\|$. For $k = 0$ we get $\|T_{12}\| = \|V_1\|$ which yields the bound on $\|T_{12}^{k+1}\|$ in (3.5).

The equality $\|P_1^{(2)} P_m^{(1)}\| = \|Z_{11}^{(h)} C_h \Pi^{(2)} Z_{mm}^{(H)} B_H \Pi^{(1)}\|$ in (3.6) follows directly from Lemma 3.2. □

So far our analysis considered general nonsingular blocks $\widehat{A}_H$ and $\widehat{A}_h$ in the matrix $\mathcal{A}$ in (1.2), and by combining (2.3) and (3.5) gives a general error bound for the multiplicative Schwarz method in terms of certain blocks of $\mathcal{A}$ and the inverses of $\widehat{A}_H$ and $\widehat{A}_h$. Note that, using the submultiplicativity of the matrix norm $\|\cdot\|$, which at this point is still a general induced norm, both convergence factors ρ_{12} and ρ_{21} can be bounded by

$$\rho_{ij} \le \|Z_{11}^{(h)} C_h\| \, \|Z_{mm}^{(H)} B_H\| \, \|\Pi^{(1)}\| \, \|\Pi^{(2)}\|. \tag{3.7}$$

In order to derive a quantitative error bound from the terms on the right-hand side, we have to make additional assumptions on $\widehat{A}_H$ and $\widehat{A}_h$. One possible choice of such assumptions is considered in the next section.

4. Error bounds for the block tridiagonal case.

We are mostly interested in the analysis of the multiplicative Schwarz method for linear algebraic systems that arise in certain discretizations of partial differential equations, and we will therefore consider

$$\widehat{A}_H = \text{tridiag}(C_H, A_H, B_H) \quad \text{and} \quad \widehat{A}_h = \text{tridiag}(C_h, A_h, B_h). \tag{4.1}$$

Additionally, we will assume that the matrices

$$A_H, B_H, C_H, A, B, C, A_h, B_h, C_h \in \mathbb{R}^{N \times N} \text{ are nonsingular,} \tag{4.2}$$

and that the matrix $\mathcal{A}$ is *row block diagonally dominant* in the sense of [3, Definition 2.1], i.e., that

$$\begin{aligned} \|A_H^{-1} B_H\| + \|A_H^{-1} C_H\| &\le 1, \\ \|A^{-1} B\| + \|A^{-1} C\| &\le 1, \\ \|A_h^{-1} B_h\| + \|A_h^{-1} C_h\| &\le 1. \end{aligned} \tag{4.3}$$

Note that, because of (4.2), each of the norms on the left-hand sides of these inequalities is *strictly* less than one.

Both A_H and A_h satisfy all assumptions of [3, Theorem 2.6]. A minor modification of the first equation in the proof of that theorem (namely multiplying both sides of this equation by C_h or B_H before taking norms) shows that the blocks of the inverses, i.e., $\widehat{A}_H^{-1} = [Z_{ij}^{(H)}]$ and $\widehat{A}_h^{-1} = [Z_{ij}^{(h)}]$, satisfy

$$\|Z_{i1}^{(h)} C_h\| \le \|Z_{11}^{(h)} C_h\| \quad \text{and} \quad \|Z_{im}^{(H)} B_H\| \le \|Z_{mm}^{(H)} B_H\|, \quad i = 1, \dots, m. \tag{4.4}$$

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Moreover, as shown in the proof of that theorem, the equations

$$Z_{11}^{(h)} = (A_h - B_h M_h)^{-1} = (I - A_h^{-1} B_h M_h)^{-1} A_h^{-1},$$
$$Z_{mm}^{(H)} = (A_H - C_H L_H)^{-1} = (I - A_H^{-1} C_H L_H)^{-1} A_H^{-1},$$

hold for some matrices $M_h, L_H \in \mathbb{R}^{N\times N}$ with $\|M_h\| \le 1$ and $\|L_H\| \le 1$; see [3, equation (2.20)]. The precise definition of M_h and L_H is not important here.

The four matrices that appear on the right-hand side of (3.7) are now given by

$$Z_{11}^{(h)} C_h = (I - A_h^{-1} B_h M_h)^{-1} A_h^{-1} C_h,$$
$$Z_{mm}^{(H)} B_H = (I - A_H^{-1} C_H L_H)^{-1} A_H^{-1} B_H,$$
$$\Pi^{(2)} = (I - A^{-1} B Z_{11}^{(h)} C_h)^{-1} A^{-1} C,$$
$$\Pi^{(1)} = (I - A^{-1} C Z_{mm}^{(H)} B_H)^{-1} A^{-1} B.$$

For any induced matrix norm $\|\cdot\|$ and any square matrix $\mathcal{M}$ with $\|\mathcal{M}\| < 1$ we have

$$\|(I - \mathcal{M})^{-1}\| \le \frac{1}{1 - \|\mathcal{M}\|};$$

see, e.g., [8, p. 351]. From $\|A_h^{-1} B_h M_h\| \le \|A_h^{-1} B_h\| < 1$ we therefore obtain

$$\|(I - A_h^{-1} B_h M_h)^{-1}\| \le \frac{1}{1 - \|A_h^{-1} B_h\|}.$$

Similarly, $\|A_H^{-1} C_H L_H\| \le \|A_H^{-1} C_H\| < 1$ implies that

$$\|(I - A_H^{-1} C_H L_H)^{-1}\| \le \frac{1}{1 - \|A_H^{-1} C_H\|},$$

and hence

$$\|Z_{11}^{(h)} C_h\| \le \frac{\|A_h^{-1} C_h\|}{1 - \|A_h^{-1} B_h\|} \equiv \eta_h \le 1, \tag{4.5}$$

$$\|Z_{mm}^{(H)} B_H\| \le \frac{\|A_H^{-1} B_H\|}{1 - \|A_H^{-1} C_H\|} \equiv \eta_H \le 1. \tag{4.6}$$

Using (4.5) and (4.6) yields

$$\|A^{-1} B Z_{11}^{(h)} C_h\| \le \eta_h \|A^{-1} B\| < 1 \quad \text{and} \quad \|A^{-1} C Z_{mm}^{(H)} B_H\| \le \eta_H \|A^{-1} C\| < 1,$$

so that

$$\|\Pi^{(2)}\| \le \frac{\|A^{-1} C\|}{1 - \eta_h \|A^{-1} B\|} \le 1 \quad \text{and} \quad \|\Pi^{(1)}\| \le \frac{\|A^{-1} B\|}{1 - \eta_H \|A^{-1} C\|} \le 1. \tag{4.7}$$

In summary, we have the following result.

LEMMA 4.1. *In the notation established above, the convergence factors of the multiplicative Schwarz method satisfy*

$$\rho_{ij} \le \frac{\eta_h \|A^{-1} C\|}{1 - \eta_h \|A^{-1} B\|} \, \frac{\eta_H \|A^{-1} B\|}{1 - \eta_H \|A^{-1} C\|}, \tag{4.8}$$

where each of the factors on the right-hand side is less than or equal to one.

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Let us illustrate the bound from Lemma 4.1 on a simple example.

EXAMPLE 4.2. Let $m \geq 1$ be given, $N = 2m + 1$, and consider the matrix

$$\mathcal{A} \equiv \text{tridiag}(-I, W, -I) \in \mathbb{R}^{N(2m+1)\times N(2m+1)},$$

where $I \in \mathbb{R}^{N\times N}$ and

$$W \equiv \text{tridiag}(-1, 4, -1) = 4\left(I - \frac{1}{4}S\right) \in \mathbb{R}^{N\times N}, \quad S \equiv \text{tridiag}(1, 0, 1) \in \mathbb{R}^{N\times N}.$$

It is well known that $\mathcal{A}$ is the result of a standard finite difference discretization of the 2D Poisson equation on the unit square and with Dirichlet boundary conditions. In our notation, $\mathcal{A}$ is of the form (1.2) and (4.1) with

$$\widehat{A}_H = \widehat{A}_h = \text{tridiag}(-I, W, -I) \in \mathbb{R}^{Nm\times Nm},$$

$B_H = B = B_h = C_H = C = C_h = -I$, $A_H = A = A_h = W$, and

$$H = h = \frac{1}{N+1}.$$

Let us first investigate row block diagonal dominance of $\mathcal{A}$ with respect to the matrix 2-norm and ∞-norm. It both cases it holds that $\|S\| \leq 2$, since $\|S\|_2 \leq \|S\|_\infty = 2$. By expanding W^{-1} using the Neumann series we get the bound

$$\|W^{-1}\| = \frac{1}{4}\left\|\sum_{k=0}^{\infty}\left(\frac{1}{4}S\right)^k\right\| \leq \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{\|S\|}{4}\right)^k = \frac{1}{4-\|S\|}, \tag{4.9}$$

and, therefore, $\|W^{-1}\| \leq \frac{1}{2}$.

For the ∞-norm the inequality in (4.9) is strict since $\|S^k\|_\infty < \|S\|_\infty^k$ for $k > N/2$, and hence

$$\|W^{-1}\|_2 \leq \|W^{-1}\|_\infty < \frac{1}{4-\|S\|_\infty} = \frac{1}{2}.$$

As a consequence, $\mathcal{A}$ is strictly row block diagonally dominant with respect to both considered matrix norms; see the conditions (4.3). Note that $\mathcal{A}$ is only weakly row diagonally dominant in the classical (scalar) sense.

Let us now concentrate on bounding the convergence factor ρ_{ij} of the multiplicative Schwarz method; see (4.8). Using the definitions (4.5) and (4.6) we obtain

$$\eta_h = \eta_H = \frac{\|W^{-1}\|}{1-\|W^{-1}\|},$$

and (4.8) yields

$$\rho_{ij} \leq \left(\frac{\frac{\|W^{-1}\|}{1-\|W^{-1}\|}\|W^{-1}\|}{1-\frac{\|W^{-1}\|}{1-\|W^{-1}\|}\|W^{-1}\|}\right)^2.$$

Since this bound is an increasing function of $\|W^{-1}\|$ in $(0, 1)$, we can use (4.9) to obtain

$$\rho_{ij} \leq \left(\frac{1}{11 - 7\|S\| + \|S\|^2}\right)^2. \tag{4.10}$$

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For the ∞-norm, the inequality in (4.10) is strict and the right-hand side is exactly one. For the 2-norm, the eigenvalues of S are known explicitly, it holds that

$$\|S\|_2 = 2 - 2\sin^2\left(\frac{\pi h}{2}\right)$$

and, therefore,

$$\rho_{ij} \leq \left(\frac{1}{1 + 6\sin^2\left(\frac{\pi h}{2}\right) + 4\sin^4\left(\frac{\pi h}{2}\right)}\right)^2 \approx \frac{1}{1 + 3\pi^2 h^2}$$

for small values of h. Thus, the convergence factor of the multiplicative Schwarz method for both considered norms is less than one, regardless of the choice of m. But note that for $N \to \infty$, and hence $h \to 0$, we can in both cases expect that $\rho_{ij} \to 1$.

In order to bound the error norm of the multiplicative Schwarz method, it remains to bound $\|T_{ij}\|$; see (2.3), (3.5), and (4.8). Let us first note that, because of the equivalence of matrix norms, there exists a constant c such that

$$\|T_{ij}\| \leq c\,\|T_{ij}\|_\infty,$$

where c can depend on the size of T_{ij}.

Now we bound $\|T_{ij}\|_\infty$. From (3.2) and (3.3) we see that

$$\|T_{12}\|_\infty = \max\{\|P^{(1)}\|_\infty,\ \|\Pi^{(2)}P_m^{(1)}\|_\infty,\ \|P^{(2)}P_m^{(1)}\|_\infty\}, \tag{4.11}$$

$$\|T_{21}\|_\infty = \max\{\|P^{(2)}\|_\infty,\ \|\Pi^{(1)}P_1^{(2)}\|_\infty,\ \|P^{(1)}P_1^{(2)}\|_\infty\}, \tag{4.12}$$

and Lemma 3.2 yields

$$\begin{aligned} P^{(1)} &= -Z_{1:m,m}^{(H)}B_H\Pi^{(1)}, & P^{(2)} &= -Z_{1:m,1}^{(h)}C_h\Pi^{(2)},\\ \Pi^{(2)}P_m^{(1)} &= -\Pi^{(2)}Z_{mm}^{(H)}B_H\Pi^{(1)}, & \Pi^{(1)}P_1^{(2)} &= -\Pi^{(1)}Z_{11}^{(h)}C_h\Pi^{(2)},\\ P^{(2)}P_m^{(1)} &= Z_{1:m,1}^{(h)}C_h\Pi^{(2)}Z_{mm}^{(H)}B_H\Pi^{(1)}, & P^{(1)}P_1^{(2)} &= Z_{1:m,m}^{(H)}B_H\Pi^{(1)}Z_{11}^{(h)}C_h\Pi^{(2)}. \end{aligned}$$

Using (4.4) we can bound the ∞-norms of these matrices as follows:

$$\begin{aligned} \|P^{(1)}\|_\infty &= \max\{\|Z_{1m}^{(H)}B_H\Pi^{(1)}\|_\infty, \ldots, \|Z_{mm}^{(H)}B_H\Pi^{(1)}\|_\infty\}\\ &\leq \|Z_{mm}^{(H)}B_H\|_\infty\|\Pi^{(1)}\|_\infty,\\ \|\Pi^{(2)}P_m^{(1)}\|_\infty &= \|\Pi^{(2)}Z_{mm}^{(H)}B_H\Pi^{(1)}\|_\infty\\ &\leq \|Z_{mm}^{(H)}B_H\|_\infty\|\Pi^{(1)}\|_\infty\|\Pi^{(2)}\|_\infty,\\ \|P^{(2)}P_m^{(1)}\|_\infty &= \max\{\|Z_{11}^{(h)}C_h\Pi^{(2)}Z_{mm}^{(H)}B_H\Pi^{(1)}\|_\infty, \ldots, \|Z_{m1}^{(h)}C_h\Pi^{(2)}Z_{mm}^{(H)}B_H\Pi^{(1)}\|_\infty\}\\ &\leq \|Z_{mm}^{(H)}B_H\|_\infty\|\Pi^{(1)}\|_\infty\|Z_{11}^{(h)}C_h\|_\infty\|\Pi^{(2)}\|_\infty, \end{aligned}$$

and

$$\begin{aligned} \|P^{(2)}\|_\infty &= \max\{\|Z_{11}^{(h)}C_h\Pi^{(2)}\|_\infty, \ldots, \|Z_{m1}^{(h)}C_h\Pi^{(2)}\|_\infty\}\\ &\leq \|Z_{11}^{(h)}C_h\|_\infty\|\Pi^{(2)}\|_\infty,\\ \|\Pi^{(1)}P_1^{(2)}\| &= \|\Pi^{(1)}Z_{11}^{(h)}C_h\Pi^{(2)}\|_\infty\\ &\leq \|Z_{11}^{(h)}C_h\|_\infty\|\Pi^{(2)}\|_\infty\|\Pi^{(1)}\|_\infty,\\ \|P^{(1)}P_1^{(2)}\|_\infty &= \max\{\|Z_{1m}^{(H)}B_H\Pi^{(1)}Z_{11}^{(h)}C_h\Pi^{(2)}\|_\infty, \ldots, \|Z_{mm}^{(H)}B_H\Pi^{(1)}Z_{11}^{(h)}C_h\Pi^{(2)}\|_\infty\}\\ &\leq \|Z_{mm}^{(H)}B_H\|_\infty\|\Pi^{(1)}\|_\infty\|Z_{11}^{(h)}C_h\|_\infty\|\Pi^{(2)}\|_\infty. \end{aligned}$$

The individual terms on the right-hand sides of the previous inequalities are all less than or equal to one. Therefore, the maximum of the first three bounds is $\|Z_{mm}^{(H)} B_H\|_\infty \|\Pi^{(1)}\|_\infty$, and the maximum of the second three bounds is $\|Z_{11}^{(h)} C_h\|_\infty \|\Pi^{(2)}\|_\infty$. Hence, using (4.11) and (4.12), and (4.5), (4.6), (4.7), we obtain

$$\|T_{12}\|_\infty \le \|Z_{mm}^{(H)} B_H\|_\infty \|\Pi^{(1)}\|_\infty \le \frac{\eta_{H,\infty}\|A^{-1}B\|_\infty}{1-\eta_{H,\infty}\|A^{-1}C\|_\infty}$$

and

$$\|T_{21}\|_\infty \le \|Z_{11}^{(h)} C_h\|_\infty \|\Pi^{(2)}\|_\infty \le \frac{\eta_{h,\infty}\|A^{-1}C\|_\infty}{1-\eta_{h,\infty}\|A^{-1}B\|_\infty},$$

where $\eta_{h,\infty}$ and $\eta_{H,\infty}$ are defined as in (4.5) and (4.6) using the ∞-norm, i.e.,

$$\eta_{h,\infty} \equiv \frac{\|A_h^{-1} C_h\|_\infty}{1-\|A_h^{-1} B_h\|_\infty}, \qquad \eta_{H,\infty} \equiv \frac{\|A_H^{-1} B_H\|_\infty}{1-\|A_H^{-1} C_H\|_\infty}. \tag{4.13}$$

Combining these bounds with Lemma 3.3 and Lemma 4.1 gives the following convergence result.

THEOREM 4.3. *Suppose that $\mathcal{A}$ as in (1.2) has blocks as in (4.1) that satisfy (4.2)–(4.3). Then the errors of the multiplicative Schwarz method (2.2) applied to the linear algebraic system (1.1) satisfy*

$$\frac{\|e^{(k+1)}\|}{\|e^{(0)}\|} \le \left(\frac{\eta_h\|A^{-1}C\|}{1-\eta_h\|A^{-1}B\|} \, \frac{\eta_H\|A^{-1}B\|}{1-\eta_H\|A^{-1}C\|} \right)^k \|T_{ij}\|, \quad k = 0, 1, 2, \ldots,$$

where η_h and η_H are defined in (4.5)–(4.6). Moreover,

$$\|T_{12}\| \le c\,\frac{\eta_{H,\infty}\|A^{-1}B\|_\infty}{1-\eta_{H,\infty}\|A^{-1}C\|_\infty}, \qquad \|T_{21}\| \le c\,\frac{\eta_{h,\infty}\|A^{-1}C\|_\infty}{1-\eta_{h,\infty}\|A^{-1}B\|_\infty},$$

where $\eta_h^{(\infty)}$ and $\eta_H^{(\infty)}$ are given by (4.13), and c is a constant such that $\|T_{ij}\| \le c\,\|T_{ij}\|_\infty$.

In many practical applications the matrices $\widehat{A}_H$ and $\widehat{A}_h$ of the form (4.1) are not only row block diagonally dominant (see (4.3)), but also *column block diagonally dominant*, i.e., they satisfy the conditions

$$\|B_H A_H^{-1}\| + \|C_H A_H^{-1}\| \le 1, \quad \|B_h A_h^{-1}\| + \|C_h A_h^{-1}\| \le 1; \tag{4.14}$$

see Definition A.1 in Appendix A. Analogously to the row block diagonally dominant case described above, a proof of Theorem A.2 along the lines of the proof of [3, Theorem 2.6] shows that if $\widehat{A}_H$ and $\widehat{A}_h$ satisfy the conditions (4.14). Then

$$Z_{11}^{(h)} = (A_h - \tilde{L}_h C_h)^{-1} = A_h^{-1}(I - \tilde{L}_h C_h A_h^{-1})^{-1},$$
$$Z_{mm}^{(H)} = (A_H - \tilde{M}_H B_H)^{-1} = A_H^{-1}(I - \tilde{M}_H B_H A_H^{-1})^{-1},$$

for some matrices $\tilde{L}_h, \tilde{M}_H \in \mathbb{R}^{N\times N}$ with $\|\tilde{L}_h\| \le 1$ and $\|\tilde{M}_H\| \le 1$. Analogously, using the Neumann series we obtain the bounds

$$\|Z_{11}^{(h)}\| \le \frac{\|A_h^{-1}\|}{1-\|C_h A_h^{-1}\|}, \qquad \|Z_{mm}^{(H)}\| \le \frac{\|A_H^{-1}\|}{1-\|B_H A_H^{-1}\|},$$

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so that

$$\|Z_{11}^{(h)}C_h\| \leq \frac{\|A_h^{-1}\|\|C_h\|}{1-\|C_hA_h^{-1}\|}, \qquad \|Z_{mm}^{(H)}B_H\| \leq \frac{\|A_H^{-1}\|\|B_H\|}{1-\|B_HA_H^{-1}\|}.$$

Therefore, if $\widehat{A}_H$ and $\widehat{A}_h$ satisfy the conditions (4.3) *and* (4.14), then

$$\|Z_{11}^{(h)}C_h\| \leq \min\left\{\frac{\|A_h^{-1}C_h\|}{1-\|A_h^{-1}B_h\|}, \frac{\|A_h^{-1}\|\|C_h\|}{1-\|C_hA_h^{-1}\|}\right\} \equiv \eta_h^{\min}, \tag{4.15}$$

and

$$\|Z_{mm}^{(H)}B_H\| \leq \min\left\{\frac{\|A_H^{-1}B_H\|}{1-\|A_H^{-1}C_H\|}, \frac{\|A_H^{-1}\|\|B_H\|}{1-\|B_HA_H^{-1}\|}\right\} \equiv \eta_H^{\min}. \tag{4.16}$$

The value of $\eta_H^{\min}$ can be much smaller than η_H, for example if $\|B_H\| \ll \|C_H\|$. Since we only improved bounds (4.5) and (4.6) on $\|Z_{11}^{(h)}C_h\|$ and $\|Z_{mm}^{(H)}B_H\|$, we can formulate a version of Theorem 4.3 where we just replace η_h and η_H with $\eta_h^{\min}$ and $\eta_H^{\min}$.

THEOREM 4.4. *Suppose that $\mathcal{A}$ as in* (1.2) *has blocks as in* (4.1) *that satisfy* (4.2)*,* (4.3)*, and* (4.14)*. Then the errors of the multiplicative Schwarz method* (2.2) *applied to the linear algebraic system* (1.1) *satisfy*

$$\frac{\|e^{(k+1)}\|}{\|e^{(0)}\|} \leq \left(\frac{\eta_h^{\min}\|A^{-1}C\|}{1-\eta_h^{\min}\|A^{-1}B\|}\,\frac{\eta_H^{\min}\|A^{-1}B\|}{1-\eta_H^{\min}\|A^{-1}C\|}\right)^k \|T_{ij}\|, \quad k=0,1,2,\ldots,$$

where $\eta_h^{\min}$ and $\eta_H^{\min}$ are defined in (4.15) *and* (4.16)*. Moreover,*

$$\|T_{12}\| \leq c\,\frac{\eta_{H,\infty}^{\min}\|A^{-1}B\|_\infty}{1-\eta_{H,\infty}^{\min}\|A^{-1}C\|_\infty}, \qquad \|T_{21}\| \leq c\,\frac{\eta_{h,\infty}^{\min}\|A^{-1}C\|_\infty}{1-\eta_{h,\infty}^{\min}\|A^{-1}B\|_\infty},$$

where $\eta_{h,\infty}^{\min}$ and $\eta_{H,\infty}^{\min}$ are given by (4.15) *and* (4.16) *using the ∞-norm, and c is a constant such that $\|T_{ij}\| \leq c\,\|T_{ij}\|_\infty$.*

In the next section we will apply these results to the study of a model problem that can be considered a two-dimensional generalization of the one-dimensional problem studied in [4].

5. Application to a discretized convection-diffusion equation.

We consider the following convection-diffusion model problem with Dirichlet boundary conditions,

$$-\epsilon\Delta u + u_y + \beta u = f \text{ in } \Omega = (0,1)\times(0,1), \qquad u = g \text{ on } \partial\Omega. \tag{5.1}$$

Here the scalar-valued function $u(x,y)$ represents the concentration of a transported quantity, $\epsilon > 0$ is the (scalar) diffusion parameter, and $\beta \geq 0$ is the (scalar) reaction parameter. We assume that the problem is *convection dominated*, i.e., that $\epsilon \ll 1$. Moreover, we assume that the parameters of the problem are chosen so that the solution $u(x,y)$ has one boundary layer at $y=1$. A common approach for discretizing such a problem is to use Shishkin meshes. This technique has been described in detail, for example, in the articles [17, Section 5] and [10], as well as in the book [13]. We therefore only briefly summarize the facts that are relevant for our analysis.

The main idea of the Shishkin mesh is to use piecewise equidistant meshes in each part of the domain, and to resolve the boundary layer by using a finer mesh close to $y=1$. In our example the *transition point* at which the mesh changes from coarse to fine in the y-direction is defined by

$$1-\tau_y, \quad \text{where} \quad \tau_y \equiv \min\left\{\frac{1}{2}, 2\epsilon\ln M\right\},$$

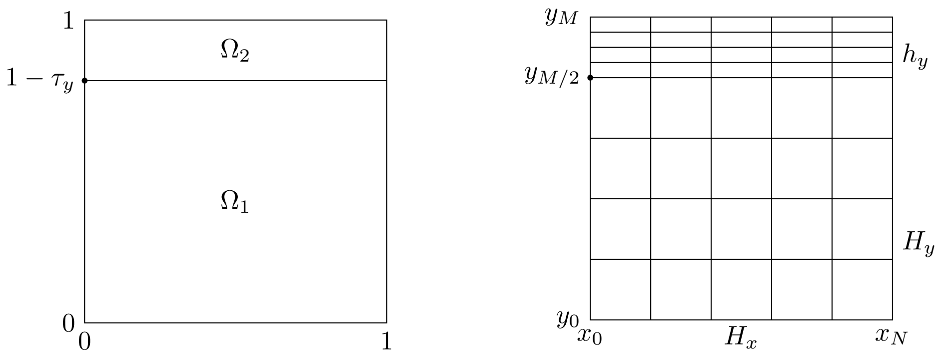

FIG. 5.1. *Division of the domain and Shishkin mesh for the problem* (5.1) *with one boundary layer at* $y = 1$.

and where the even integer $M \geq 4$ denotes the number of mesh intervals in the y-direction. The assumption $\epsilon \ll 1$ implies that $\tau_y \ll 1$, so that $1 - \tau_y$ will be very close to $y = 1$, unless M is chosen extremely large. The domain $\overline{\Omega}$ is thus decomposed into the two overlapping subdomains

$$\Omega_1 = [0,1] \times [0, 1-\tau_y] \quad \text{and} \quad \Omega_2 = [0,1] \times [1-\tau_y, 1];$$

see the left part of Figure 5.1.

Let the integer $N \geq 3$ denote the number of equidistant intervals used in the x-direction; see the right part of Figure 5.1. In the literature on Shishkin mesh discretizations (including the references cited above) we usually find $N = M$, but being able to choose $N \neq M$ gives some additional flexibility in the construction of the mesh.

We denote the mesh width in the x-direction by H_x, and the widths before and after the transition point in the y-direction by H_y and h_y, i.e.,

$$H_x \equiv \frac{1}{N}, \quad H_y \equiv \frac{2(1-\tau_y)}{M}, \quad h_y \equiv \frac{2\tau_y}{M}.$$

Then the nodes of the Shishkin mesh are given by

$$\{(iH_x, y_j) \,:\, i = 0,\dots,N; j = 0,\dots,M\} \subset \overline{\Omega}, \quad \text{where} \tag{5.2}$$
$$y_j \equiv \begin{cases} jH_y & \text{for } j = 0,\dots,M/2, \\ 1-(M-j)h_y & \text{for } j = M/2+1,\dots,M. \end{cases}$$

The ratio between the different mesh sizes in the y-direction is given by

$$\frac{h_y}{H_y} = \frac{\tau_y}{1-\tau_y} = \tau_y + \mathcal{O}(\tau_y^2) \ll 1.$$

Using the standard upwind finite difference discretization of (5.1) and the lexicographical line ordering of the unknowns yields a linear algebraic system with $\mathcal{A}$ as in (1.2) and the submatrices $\widehat{A}_H$ and $\widehat{A}_h$ having the block tridiagonal structure (4.1). Each block row of $\mathcal{A}$ corresponds to one row of unknowns in the mesh, and hence each block has size $(N-1) \times (N-1)$; cf. the right part of Figure 5.1. Note that, due to the standard notation in the context of Shishkin mesh discretizations, we here slightly depart from the notation in (1.2) and (4.1), where the individual blocks have size $N \times N$. In the y-direction there are $M - 1$

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interior nodes, and one of them is at the transition point $y_{M/2} = 1 - \tau_y$. In the notation of (1.2), we have $m = M/2 - 1$, and $\mathcal{A}$ has size $(N-1)(M-1) \times (N-1)(M-1)$.

Following the description in [12, 17], we see that the (nonzero) off-diagonal blocks of $\mathcal{A}$ are given by

$$C_H = d_H I, \quad C = dI, \quad C_h = d_h I, \quad B_H = e_H I, \quad B = eI, \quad B_h = e_h I,$$

where

$$d_H = -\frac{\epsilon}{H_y^2} - \frac{1}{H_y}, \qquad d = -\frac{2\epsilon}{H_y(H_y + h_y)} - \frac{1}{H_y}, \qquad d_h = -\frac{\epsilon}{h_y^2} - \frac{1}{h_y},$$
$$e_H = -\frac{\epsilon}{H_y^2}, \qquad e = -\frac{2\epsilon}{h_y(H_y + h_y)}, \qquad e_h = -\frac{\epsilon}{h_y^2}, \tag{5.3}$$

and $A_H = \text{tridiag}(c_H, a_H, b_H)$, $A = \text{tridiag}(c, a, b)$, $A_h = \text{tridiag}(c_h, a_h, b_h)$, where

$$c_H = -\frac{\epsilon}{H_x^2}, \qquad a_H = \frac{2\epsilon}{H_x^2} + \frac{2\epsilon}{H_y^2} + \frac{1}{H_y} + \beta, \qquad b_H = -\frac{\epsilon}{H_x^2},$$
$$c = -\frac{\epsilon}{H_x^2}, \qquad a = \frac{2\epsilon}{H_x^2} + \frac{2\epsilon}{H_y h_y} + \frac{1}{H_y} + \beta, \qquad b = -\frac{\epsilon}{H_x^2},$$
$$c_h = -\frac{\epsilon}{H_x^2}, \qquad a_h = \frac{2\epsilon}{H_x^2} + \frac{2\epsilon}{h_y^2} + \frac{1}{h_y} + \beta, \qquad b_h = -\frac{\epsilon}{H_x^2}.$$

We will now show that for this model problem the assumptions of Theorem 4.4 are satisfied.

LEMMA 5.1. *All nonzero blocks of the matrix $\mathcal{A}$ described above are nonsingular. Moreover, for the matrix ∞-norm the matrix $\mathcal{A}$ satisfies the conditions (4.3), i.e., it is row block diagonally dominant, and the submatrices $\widehat{A}_H$ and $\widehat{A}_h$ satisfy the conditions (4.14), i.e., they are column block diagonally dominant.*

Proof. Note that all (nonzero) off-diagonal entries of $\mathcal{A}$ are negative, and that the diagonal entries a_H, a, a_h are positive. Moreover,

$$a_H + b_H + c_H + d_H + e_H = a + b + c + d + e = a_h + b_h + c_h + d_h + e_h = \beta \geq 0.$$

It is thus easy to see that all nonzero blocks of $\mathcal{A}$ are nonsingular. In particular, the off-diagonal blocks are nonzero multiples of the identity matrix and, using the above inequality, the diagonal blocks are strictly diagonally dominant.

To prove (4.3) and (4.14) for the ∞-norm, we just need to show that

$$|e_H + d_H| \|A_H^{-1}\|_\infty \leq 1, \quad |e + d| \|A^{-1}\|_\infty \leq 1, \quad |e_h + d_h| \|A_h^{-1}\|_\infty \leq 1, \tag{5.4}$$

and hence we need to bound the ∞-norms of matrices A_H^{-1}, A^{-1}, and A_h^{-1}.

First note that for any nonsingular matrix $\mathcal{M}$ and an induced matrix norm we have

$$\|\mathcal{M}^{-1}\| = \max_{\|v\|=1} \left\| \mathcal{M}^{-1} \left(\frac{\mathcal{M}v}{\|\mathcal{M}v\|} \right) \right\| = \frac{1}{\min_{\|v\|=1} \|\mathcal{M}v\|}.$$

Therefore, if $\|\mathcal{M}v\| \geq \gamma > 0$ for any unit norm vector v, then $\|\mathcal{M}^{-1}\| \leq \gamma^{-1}$.

Second, suppose that $\mathcal{M}$ is a strictly diagonally dominant tridiagonal Toeplitz matrix $\mathcal{M} = \text{tridiag}(\hat{c}, \hat{a}, \hat{b})$, where $\hat{a} > 0$, $\hat{b} < 0$, $\hat{c} < 0$, and $\hat{a} + \hat{b} + \hat{c} > 0$. We would like to bound $\|\mathcal{M}v\|_\infty$ for any unit norm vector v from below. If $\|v\|_\infty = 1$, then there is an index i such

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that $|v_i| = 1$. Without loss of generality we can assume that $v_i = 1$, because changing the sign of the vector does not change $\|\mathcal{M}v\|_\infty$. Defining $v_0 = 0$ and $v_{n+1} = 0$ we obtain

$$\|\mathcal{M}v\|_\infty \geq |v_{i-1}\hat{c} + \hat{a} + v_{i+1}\hat{b}| \geq \hat{a} + \hat{b} + \hat{c},$$

and therefore

$$\|\mathcal{M}^{-1}\|_\infty \leq \frac{1}{\hat{a} + \hat{b} + \hat{c}}. \tag{5.5}$$

In order to prove (5.4), we now apply the bound (5.5) to matrices A_H, A, and A_h, which are strictly diagonally dominant tridiagonal Toeplitz matrices with the required sign pattern. For A_H we get

$$|e_H + d_H|\|A_H^{-1}\|_\infty \leq \frac{|e_H + d_H|}{a_H + b_H + c_H} = \frac{|e_H + d_H|}{|e_H + d_H| + \beta} \leq 1,$$

and the other inequalities in (5.4) follow analogously. □

Lemma 5.1 ensures that the assumptions of Theorem 4.4 are satisfied for matrices arising from the discretization of the problem (5.1) that we have described above (namely, upwind differences on a Shishkin mesh). We therefore obtain the following convergence result for the multiplicative Schwarz method.

COROLLARY 5.2. *Consider the linear algebraic system (1.1) arising from the upwind discretization of the boundary value problem (5.1) on the Shishkin mesh (5.2). Then the errors of the multiplicative Schwarz method (2.2) applied to this system satisfy*

$$\frac{\|e^{(k+1)}\|_\infty}{\|e^{(0)}\|_\infty} \leq \rho^k \|T_{ij}\|_\infty, \quad k = 0, 1, 2, \ldots, \tag{5.6}$$

where

$$\rho \equiv \frac{\epsilon}{\epsilon + H_y}, \qquad \|T_{12}\|_\infty \leq \rho \quad \textit{and} \quad \|T_{21}\|_\infty \leq 1. \tag{5.7}$$

Proof. To prove the bounds (5.6)–(5.7) we apply Theorem 4.4 with the ∞-norm and bound the factors $\eta_{h,\infty}^{\min}$ and $\eta_{H,\infty}^{\min}$ that correspond to the discretization scheme (5.3).

Using $|d_h| > |e_h|$ we get

$$\eta_{h,\infty}^{\min} = \min\left\{\frac{|d_h|\|A_h^{-1}\|_\infty}{1 - |e_h|\|A_h^{-1}\|_\infty}, \frac{|d_h|\|A_h^{-1}\|_\infty}{1 - |d_h|\|A_h^{-1}\|_\infty}\right\} = \frac{|d_h|\|A_h^{-1}\|_\infty}{1 - |e_h|\|A_h^{-1}\|_\infty} \leq 1.$$

Similarly, from $|d_H| > |e_H|$ it follows that

$$\eta_{H,\infty}^{\min} = \min\left\{\frac{|e_H|\|A_H^{-1}\|_\infty}{1 - |d_H|\|A_H^{-1}\|_\infty}, \frac{|e_H|\|A_H^{-1}\|_\infty}{1 - |e_H|\|A_H^{-1}\|_\infty}\right\} = \frac{|e_H|\|A_H^{-1}\|_\infty}{1 - |e_H|\|A_H^{-1}\|_\infty} \leq \left|\frac{e_H}{d_H}\right|.$$

Hence, an upper bound on ρ_{12} and ρ_{21} is given by

$$\frac{\eta_{h,\infty}^{\min}\|A^{-1}C\|_\infty}{1 - \eta_{h,\infty}^{\min}\|A^{-1}B\|_\infty} \frac{\eta_{H,\infty}^{\min}\|A^{-1}B\|_\infty}{1 - \eta_{H,\infty}^{\min}\|A^{-1}C\|_\infty} \leq \eta_{h,\infty}^{\min}\eta_{H,\infty}^{\min} \leq \left|\frac{e_H}{d_H}\right| \equiv \rho.$$

Finally,

$$\|T_{12}\|_\infty \leq \eta_{H,\infty} \leq \left|\frac{e_H}{d_H}\right|, \qquad \|T_{21}\|_\infty \leq \eta_{h,\infty}^{\min} \leq 1,$$

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and substituting the values of e_H and d_H given in (5.3) yields the desired result. □

Note that the bound (5.6) does not depend on the choice of N, i.e., the size of the mesh in the x-direction. Moreover, for a fixed choice of M, and hence of H_y, the value of $\epsilon/(\epsilon + H_y)$ decreases with decreasing ϵ. Similarly to the one-dimensional model problem studied in [4], this indicates a faster convergence of the multiplicative Schwarz method for smaller ϵ, meaning larger convection-dominance, which is confirmed in the following numerical example.

EXAMPLE 5.3. We consider the model problem (5.1) with $\beta = 0$, $f = 0$, and boundary conditions determined by the function

$$u(x, y) = (2x - 1)\left(\frac{1 - e^{(y-1)/\epsilon}}{1 - e^{-1/\epsilon}}\right), \tag{5.8}$$

which is a minor variation of the model problem in [5, Example 6.1.1]. All computations in this example were done in MATLAB R2019a. Figure 5.2 shows $u(x, y)$ with $\epsilon = 0.01$ on $\Omega = (0, 1) \times (0, 1)$, and we clearly see the boundary layer close to $y = 1$, even for this relatively large value of ϵ.

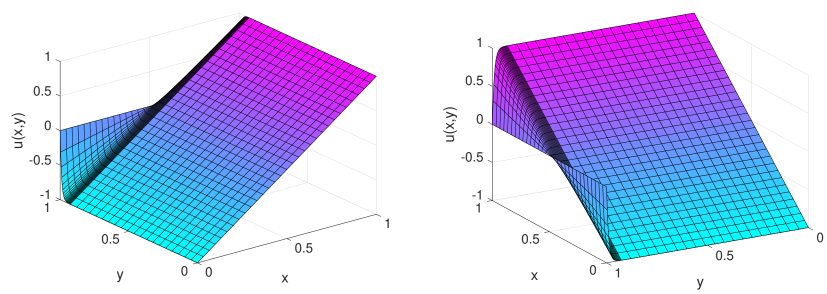

FIG. 5.2. *The function* (5.8) *with* $\epsilon = 0.01$ *on* $\Omega = (0, 1) \times (0, 1)$ *viewed from two different angles.*

We discretize the problem using upwind differences on a Shishkin mesh as described above with $N = 30$ and $M = 40$, and thus obtain a linear algebraic system $\mathcal{A}x = b$ with $\mathcal{A}$ of size 1131×1131. We compute (an approximation to) the exact solution $x = \mathcal{A}^{-1}b$ using MATLAB's backslash operator, and apply the multiplicative Schwarz method with $x^{(0)} = 0$. Figure 5.3 shows the resulting relative error norms

$$\frac{\|x - x^{(k)}\|_\infty}{\|x\|_\infty}, \ k = 0, 1, 2, \ldots$$

for the two iteration matrices T_{12} and T_{21} (solid lines) and the corresponding upper bounds from Corollary 5.2 for $\epsilon = 10^{-4}$ (left) and $\epsilon = 10^{-8}$ (right). In all cases the bounds are very close to the actual error norms, and, as indicated above, the multiplicative Schwarz method converges faster for problems that are more convection-dominated.

In Table 5.1 we show, for different N, M and ϵ, the values of the convergence factor ρ_{12} as given in (3.6) with $\|\cdot\| = \|\cdot\|_\infty$, and the value of ρ in (5.7), which represents an upper bound on ρ_{12}. In all cases the two values are quite close to each other.

6. Concluding discussion. Motivated by an analysis for a one-dimensional convection-diffusion model problem in [4], we have studied the convergence of the multiplicative Schwarz method for matrices with a special block structure. After deriving a general expression for the convergence factor of the method, we have focussed on block tridiagonal matrices, and

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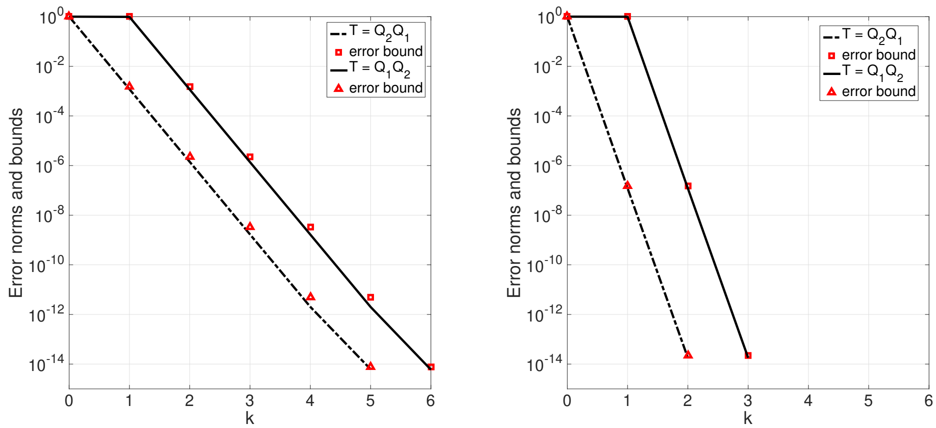

FIG. 5.3. *Convergence of multiplicative Schwarz and error bounds for* $\epsilon = 10^{-4}$ *(left)* $\epsilon = 10^{-8}$ *(right).*

applied recent results on block diagonal dominance from [3] in order to obtain quantitative error bounds that are valid from the first iteration. In our analysis we did not use any of the usual assumptions on the matrices in this context, such as symmetry, or the M- or H-matrix properties. We illustrated our bounds numerically on a two-dimensional convection-diffusion model problem that was discretized using a Shishkin mesh, and we found that the bounds are very close to the actual error norms produced by the method. We will now briefly discuss possible generalizations and alternative applications of our approach.

Variable block sizes of $\mathcal{A}$ in (1.2). The results of Sections 2–4, which are based on the algebraic structure of the matrix (1.2) only, can be generalized to matrices $\widehat{A}_H$ and $\widehat{A}_h$ having different sizes. Such a generalization is straightforward if the sizes of $\widehat{A}_H$ and $\widehat{A}_h$ are multiples of N, i.e., $\widehat{A}_H \in \mathbb{R}^{Nm\times Nm}$ and $\widehat{A}_h \in \mathbb{R}^{Ns\times Ns}$ with $s \neq m$, and more technical when this assumption is not satisfied. We have chosen $\widehat{A}_H$ and $\widehat{A}_h$ of the same size in order to reduce the technicalities in the analysis, and because our main application in Section 5 relies on the idea of the Shishkin mesh, which is composed of piecewise equidistant meshes which have the same number of unknowns.

General tridiagonal blocks in (4.1). In (4.1) we have assumed that the matrices $\widehat{A}_H$ and $\widehat{A}_h$ have a block Toeplitz structure. Apart from simplicity of notation, the main motivation behind this assumption was also the structure of the discretized problem (5.1) with constant coefficients studied in Section 5. Suppose that $\widehat{A}_H$ and $\widehat{A}_h$ have a general block tridiagonal structure of the form

$$\widehat{A}_H = \mathrm{tridiag}(C_{H,i}, A_{H,i}, B_{H,i}) \quad \text{and} \quad \widehat{A}_h = \mathrm{tridiag}(C_{h,i}, A_{h,i}, B_{h,i}) \tag{6.1}$$

with nonsingular $N \times N$ blocks $A_{H/h,i}, B_{H/h,i}, C_{H/h,i}$ for $i = 1, \ldots, m$. Such a matrix $\mathcal{A}$ can be obtained, for instance, by discretizing a boundary value problem like (5.1), but with nonconstant coefficients. An analysis of the multiplicative Schwarz method for such a matrix $\mathcal{A}$ following the approach in this paper is still possible, since the results from [3] on block diagonal dominance are formulated for general block tridiagonal matrices; see also Appendix A. A generalization of Theorem 4.4 to $\mathcal{A}$ with blocks (6.1) would require that the conditions (4.3) hold in every block row, and then, analogously to (4.5)–(4.6), every block row in $\widehat{A}_H$ or $\widehat{A}_h$ would give a parameter $\eta_{H,i}$ or $\eta_{h,i}$, respectively.

TABLE 5.1
Values of ρ_{12} computed using (3.6) with $\|\cdot\| = \|\cdot\|_\infty$ and ρ in (5.7) for different values of N, M and ϵ.

ϵ	ρ_{12} in (3.6)	ρ in (5.7)
$N=20, M=20, \mathcal{A}\in\mathbb{R}^{361\times361}$		
10^{-8}	7.5×10^{-8}	1.0×10^{-7}
10^{-6}	7.5×10^{-6}	1.0×10^{-5}
10^{-4}	7.5×10^{-4}	1.0×10^{-3}
10^{-2}	7.0×10^{-2}	9.6×10^{-2}
$N=30, M=30, \mathcal{A}\in\mathbb{R}^{841\times841}$		
10^{-8}	1.2×10^{-7}	1.5×10^{-7}
10^{-6}	1.2×10^{-5}	1.5×10^{-5}
10^{-4}	1.2×10^{-3}	1.5×10^{-3}
10^{-2}	1.1×10^{-1}	1.4×10^{-1}
$N=40, M=40, \mathcal{A}\in\mathbb{R}^{1521\times1521}$		
10^{-8}	1.7×10^{-7}	2.0×10^{-7}
10^{-6}	1.7×10^{-5}	2.0×10^{-5}
10^{-4}	1.7×10^{-3}	2.0×10^{-3}
10^{-2}	1.4×10^{-1}	1.8×10^{-1}
$N=50, M=50, \mathcal{A}\in\mathbb{R}^{2401\times2401}$		
10^{-8}	2.2×10^{-7}	2.5×10^{-7}
10^{-6}	2.2×10^{-5}	2.5×10^{-5}
10^{-4}	2.2×10^{-3}	2.5×10^{-3}
10^{-2}	1.8×10^{-1}	2.1×10^{-1}

ϵ	ρ_{12} in (3.6)	ρ in (5.7)
$N=20, M=30, \mathcal{A}\in\mathbb{R}^{551\times551}$		
10^{-8}	1.2×10^{-7}	1.5×10^{-7}
10^{-6}	1.2×10^{-5}	1.5×10^{-5}
10^{-4}	1.2×10^{-3}	1.5×10^{-3}
10^{-2}	1.1×10^{-1}	1.4×10^{-1}
$N=30, M=40, \mathcal{A}\in\mathbb{R}^{1131\times1131}$		
10^{-8}	1.7×10^{-7}	2.0×10^{-7}
10^{-6}	1.7×10^{-5}	2.0×10^{-5}
10^{-4}	1.7×10^{-3}	2.0×10^{-3}
10^{-2}	1.4×10^{-1}	1.8×10^{-1}
$N=40, M=50, \mathcal{A}\in\mathbb{R}^{1911\times1911}$		
10^{-8}	2.2×10^{-7}	2.5×10^{-7}
10^{-6}	2.2×10^{-5}	2.5×10^{-5}
10^{-4}	2.2×10^{-3}	2.5×10^{-3}
10^{-2}	1.8×10^{-1}	2.1×10^{-1}
$N=50, M=60, \mathcal{A}\in\mathbb{R}^{2891\times2891}$		
10^{-8}	2.6×10^{-7}	3.0×10^{-7}
10^{-6}	2.6×10^{-5}	3.0×10^{-5}
10^{-4}	2.6×10^{-3}	3.0×10^{-3}
10^{-2}	2.1×10^{-1}	2.5×10^{-1}

Problems with two boundary layers. More practical two-dimensional problems arise when considering convection in both directions, e.g.,

$$-\epsilon\Delta u + u_x + u_y + \beta u = f \text{ in } \Omega = (0,1)\times(0,1),\ 0<\epsilon\ll 1,\ \beta\geq 0,\quad u=0 \text{ on } \partial\Omega. \tag{6.2}$$

In this case the solution has two boundary layers at $x = 1$ and $y = 1$; for more details, see, e.g., [12, 13, 15]. One can again use an appropriate Shishkin mesh to resolve the boundary layers (see Figure 6.1) and discretize the problem using standard upwind finite differences and the lexicographical line ordering of the unknowns (see, e.g., [12]), which yields a matrix $\mathcal{A}$ with a block tridiagonal structure. The individual blocks are not Toeplitz anymore, but their more complicated "Toeplitz-like" structure can be analyzed using the chosen discretization scheme.

To solve the resulting algebraic system we can again apply the multiplicative Schwarz method. Now we have four regions Ω_{ij} (see Figure 6.1), and we can choose restriction operators corresponding to these regions. Our numerical experiments predict that the multiplicative Schwarz method converges for any ordering of subdomains, but some orderings lead to a faster convergence (e.g., Ω_{11}, Ω_{12}, Ω_{21}, Ω_{22}) than the others (e.g., Ω_{12}, Ω_{22}, Ω_{21}, Ω_{11}). In all cases the iteration matrices seem to have a low numerical rank (close to N). The algebraic structure of the iteration matrices is now more complicated, but we believe that it is still analyzable along the lines of Section 3. A full analysis is, however, beyond the scope of this paper.

Multiplicative Schwarz preconditioned GMRES. It can be seen from (2.2) that the fixed point of the multiplicative Schwarz iteration, and hence the solution of (1.1), satisfies

$$(I - T_{ij})x = v. \tag{6.3}$$

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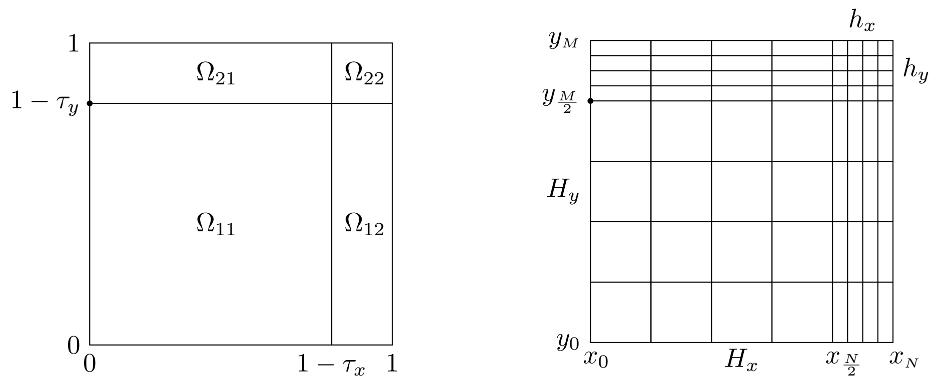

FIG. 6.1. *Division of the domain and Shishkin mesh for the problem* (6.2) *with two boundary layers.*

As discussed also in [4, Section 6], if we apply the Krylov subspace method GMRES [16] to (6.3), then the multiplicative Schwarz method can be seen as a preconditioner; see, e.g., [9]. For the matrices studied in this paper we have $\mathrm{rank}(T_{ij}) \leq N$ (see Lemma 3.1), and therefore

$$\dim\left(\mathcal{K}_k(I - T_{ij}, r_0)\right) \leq N + 1,$$

for any initial residual r_0. Consequently, the multiplicative Schwarz GMRES will converge to the solution in at most $N+1$ steps (in exact arithmetic), even when the multiplicative Schwarz method itself converges slowly or diverges. Note that T_{ij} will also have low rank for discretized boundary value problems like (5.1) with nonconstant coefficients, and possibly also for more complicated problems with several boundary layers.

Additive Schwarz method. In this paper we obtained results for the multiplicative Schwarz method solving linear algebraic systems with $\mathcal{A}$ given by (1.2). A natural question is whether analogous results can be obtained when considering the additive Schwarz method. In that method we use the iteration scheme

$$x^{(k+1)} = Tx^{(k)} + (P_1 + P_2)x, \quad T \equiv I - (P_1 + P_2);$$

see, e.g., [1, 14]. Using the structure of projection matrices P_1 and P_2 described in Section 3 we obtain

$$T = -\begin{bmatrix} 0_{N(m-1)} & & & P^{(1)}_{1:m-1} & \\ & & & P^{(1)}_{m} & \\ & \Pi^{(2)} & I_N & \Pi^{(1)} & \\ & P^{(2)}_{1} & & & \\ & P^{(2)}_{2:m} & & & 0_{N(m-1)} \end{bmatrix}.$$

Obviously, the spectral radius of T is larger than or equal to one, so that the additive Schwarz method is not convergent in general. Nevertheless, the matrix $I - T$ in the preconditioned system $(I - T)x = v$ is nonsingular, and the additive Schwarz method can be used as a preconditioner for GMRES. Since T has rank $3N$, the preconditioned GMRES will converge in at most $3N + 1$ steps. In the same spirit, the damped additive Schwarz method [14] can be analyzed.

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Appendix A. Column block diagonal dominance of matrices. Analogously to [3, Definition 2.1], we can define column block diagonal dominance of matrices as follows.

DEFINITION A.1. *A matrix of the form*

$$A = [A_{ij}] \quad \text{with blocks } A_{ij} \in \mathbb{C}^{m\times m} \text{ for } i,j = 1,\dots,n$$

is called column block diagonally dominant *(with respect to the matrix norm* $\|\cdot\|$*), when the diagonal blocks* A_{jj} *are nonsingular, and*

$$\sum_{\substack{i=1\\ i\neq j}}^{n} \|A_{ij}A_{jj}^{-1}\| \leq 1, \quad \text{for } j = 1,\dots,n. \tag{A.1}$$

If strict inequality holds in (A.1) *then* A *is called* column block strictly diagonally dominant *(with respect to the matrix norm* $\|\cdot\|$*).*

Restricting our attention to block *tridiagonal* matrices $A = \mathrm{tridiag}(C_i, A_i, B_i)$ as in [3, Equation (2.4)], and following the notation of that paper, we define

$$\tilde{\tau}_i = \frac{\|C_iA_i^{-1}\|}{1 - \|B_{i-1}A_i^{-1}\|}, \quad \tilde{\omega}_i = \frac{\|B_{i-1}A_i^{-1}\|}{1 - \|C_iA_i^{-1}\|}, \quad \text{for } i = 1,\dots,n,$$

where $B_0 = C_n = 0$. The column block diagonal dominance of A then implies that $0 \leq \tilde{\tau}_i \leq 1$ and $0 \leq \tilde{\omega}_i \leq 1$. The same approach as in [3, Section 2] now yields the the following result, which is the "column version" of [3, Theorem 2.6].

THEOREM A.2. *Let* $A = \mathrm{tridiag}(C_i, A_i, B_i)$ *be column block diagonally dominant, and suppose that the blocks* B_i*,* C_i *for* $i = 1,\dots,n-1$ *are nonsingular. If in addition*

$$\|C_1A_1^{-1}\| < 1 \quad \text{and} \quad \|B_{n-1}A_n^{-1}\| < 1,$$

then $A^{-1} = [Z_{ij}]$ *with*

$$\|Z_{ij}\| \leq \|Z_{ii}\| \prod_{k=i+1}^{j} \tilde{\omega}_k, \quad \text{for all } i < j,$$

$$\|Z_{ij}\| \leq \|Z_{ii}\| \prod_{k=j}^{i-1} \tilde{\tau}_k, \quad \text{for all } i > j.$$

Moreover, for $i = 1,\dots,n$*,*

$$\frac{\|I\|}{\|A_i\| + \tilde{\tau}_{i-1}\|B_{i-1}\| + \tilde{\omega}_{i+1}\|C_i\|} \leq \|Z_{ii}\| \leq \frac{\|I\|}{\|A_i^{-1}\|^{-1} - \tilde{\tau}_{i-1}\|B_{i-1}\| - \tilde{\omega}_{i+1}\|C_i\|},$$

provided that the denominator of the upper bound is larger than zero, and where we set $\tilde{\tau}_0 = \tilde{\omega}_{n+1} = 0$.

REFERENCES

[1] M. BENZI, A. FROMMER, R. NABBEN, AND D. B. SZYLD, *Algebraic theory of multiplicative Schwarz methods*, Numer. Math., 89 (2001), pp. 605–639.

[2] R. BRU, F. PEDROCHE, AND D. B. SZYLD, *Overlapping additive and multiplicative Schwarz iterations for H-matrices*, Linear Algebra Appl., 393 (2004), pp. 91–105.

[3] C. ECHEVERRÍA, J. LIESEN, AND R. NABBEN, *Block diagonal dominance of matrices revisited: Bounds for the norms of inverses and eigenvalue inclusion sets*, Linear Algebra Appl., 553 (2018), pp. 365–383.

[4] C. ECHEVERRÍA, J. LIESEN, D. B. SZYLD, AND P. TICHÝ, *Convergence of the multiplicative Schwarz method for singularly perturbed convection-diffusion problems discretized on a Shishkin mesh*, Electron. Trans. Numer. Anal., 48 (2018), pp. 40–62.
http://etna.ricam.oeaw.ac.at/vol.48.2018/pp40-62.dir/pp40-62.pdf

[5] H. C. ELMAN, D. J. SILVESTER, AND A. J. WATHEN, *Finite Elements and Fast Iterative Solvers: With Applications in Incompressible Fluid Dynamics*, 2nd ed., Oxford University Press, Oxford, 2014.

[6] A. FROMMER, R. NABBEN, AND D. B. SZYLD, *Convergence of stationary iterative methods for Hermitian semidefinite linear systems and applications to Schwarz methods*, SIAM J. Matrix Anal. Appl., 30 (2008), pp. 925–938.

[7] A. FROMMER AND D. B. SZYLD, *On necessary conditions for convergence of stationary iterative methods for Hermitian semidefinite linear systems*, Linear Algebra Appl., 453 (2014), pp. 192–201.

[8] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, 2nd ed., Cambridge University Press, Cambridge, 2013.

[9] G. A. A. KAHOU, E. KAMGNIA, AND B. PHILIPPE, *An explicit formulation of the multiplicative Schwarz preconditioner*, Appl. Numer. Math., 57 (2007), pp. 1197–1213.

[10] N. KOPTEVA AND E. O'RIORDAN, *Shishkin meshes in the numerical solution of singularly perturbed differential equations*, Int. J. Numer. Anal. Model., 7 (2010), pp. 393–415.

[11] D. KRATZER, S. V. PARTER, AND M. STEUERWALT, *Block splittings for the conjugate gradient method*, Comput. & Fluids, 11 (1983), pp. 255–279.

[12] T. LINSS AND M. STYNES, *Numerical methods on Shishkin meshes for linear convection-diffusion problems*, Comput. Methods Appl. Mech. Engrg., 190 (2001), pp. 3527–3542.

[13] J. J. H. MILLER, E. O'RIORDAN, AND G. I. SHISHKIN, *Fitted Numerical Methods for Singular Perturbation Problems: Error Estimates in the Maximum Norm for Linear Problems in One and Two Dimensions*, World Scientific Publishing, Hackensack, 2012.

[14] R. NABBEN AND D. B. SZYLD, *Schwarz iterations for symmetric positive semidefinite problems*, SIAM J. Matrix Anal. Appl., 29 (2006/07), pp. 98–116.

[15] H.-G. ROOS, M. STYNES, AND L. TOBISKA, *Robust Numerical Methods for Singularly Perturbed Differential Equations*, 2nd ed., Springer, Berlin, 2008.

[16] Y. SAAD AND M. H. SCHULTZ, *GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems*, SIAM J. Sci. Statist. Comput., 7 (1986), pp. 856–869.

[17] M. STYNES, *Steady-state convection-diffusion problems*, Acta Numer., 14 (2005), pp. 445–508.