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The Limits of the Standard Solar Model

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Abstract

In the standard solar model the possible distributions of the internal observables are restricted by the external properties of the sun. The remaining uncertainty with respect to alpha–particle diffusion in the solar interior can be removed by the measurement of the neutrino flux.

1. Introduction

The concept of a standard solar model (SSM) starts with the assumption that the chemical composition and the physical observables in the solar interior are determined by the radius R_s, the total mass M_s, the electromagnetic radiation power L_s of the sun, by the time τ_s of the sun on the main sequence branch of the Hertzsprung–Russell diagram, by the initial chemical composition of the solar material and by the reaction rates for various HH- and HC-cycles of hydrogen burning. If the initial mass abundances of hydrogen (^{1}H), helium (^{4}He) and of heavier elements (atomic number > 2) are denoted by X_0, Y_0 and Z_0, respectively, then the initial radial distribution of the mass density

$$\rho = \rho(x), \quad 0 < x = r/R_s < 1$$

can be written as

$$\begin{aligned} \rho_x &= X_0\rho, & \rho &= \rho_x + \rho_y + \rho_z \\ \rho_y &= Y_0\rho, & 1 &= X_0 + Y_0 + Z_0 \\ \rho_z &= Z_0\rho, & X_6 &= X(C) + X(N) < Z_0. \end{aligned} \tag{1}$$

The mass abundance X_6 of carbon plus nitrogen is part of the number Z_0. The energy production of the HC- or CNO-cycle is proportional to this quantity

$$L_s = L_{HH} + L_{HC}, \quad L_{HC} \propto X_6. \tag{2}$$

The number X_6 remains constant during the solar evolution. As an interaction between the sun and the interstellar gas is possible, the number X_6 is not necessarily the same as the corresponding abundance in the atmosphere of the sun. In any case it should not exceed this value

$$X_6 < 4 \times 10^{-3}.$$

Within the SSM the choice of the number X_6 is not free. It has to be chosen in such a way that the sum (2) agrees with the experimental value L_s of the radiation power.

In the relations (1) the contribution of the ^{2}H- and ^{3}He-nuclei shall be neglected. The hydrogen burning on the main sequence changes only the abundances X and Y of hydrogen and helium, respectively, where the total number $\tilde{N}_\alpha$ of α-particles produced within the sun is fixed by the slightly increasing radiation power L of the sun during the last 4.6 Gigayears. Thus, it is not possible to reduce the temperature T by a reduction of the local helium abundance Y

$$Y = Y_0 + \tilde{Y}, \quad \tilde{n}_\alpha = \tilde{Y} N_A \rho, \quad \tilde{N}_\alpha = \int \tilde{n}_\alpha \, dV \tag{3}$$

because the number $\tilde{N}_\alpha$ has to be kept constant. $\tilde{n}_\alpha$ is the local number density of the additional α-particles. It is not possible either to reduce Y_0 in an arbitrary way because the helium abundance 4.6 Gigayears ago cannot be smaller than the abundance in the primordial mixture and it cannot be larger than the present helium abundance in the galaxy

$$\begin{gathered} 0.23 < Y_0 < 0.28, \quad 0 < Z_0 < 0.02 \\ X_0 = 0.727, \quad Y_0 = 0.256, \quad Z_0 = 0.017. \end{gathered} \tag{4}$$

The special choice (4) shall be used in the following.

The SSM assumes energy transport by radiation in the radial region $r < 0.7R_s$. This assumption determines the opacity κ as a function of the distance r from the solar center. On the other hand the quantity κ

depends on the possible photon absorption processes at given temperature, pressure and chemical composition. A rough agreement between these two functions $\kappa(x)$ is sufficient because locally partial energy transport is possible by the convection of H- and He-gas.

A further boundary condition arises from solar seismology: The proper frequencies of the solar oscillations depend on the velocity c_s of sound

$$v_s = v_s(x) = [P(x)/\rho(x)]^{1/2}, \tag{5}$$

where the gravitation pressure P is related to the mass density ρ in the case of hydrostatic equilibrium

$$P(x) = P(0) - (G_N/R_s) \int_0^x M_y \rho(y) y^{-2} dy$$

$$P(1) = 0, \quad M_1 = M_s, \quad M_x = 4\pi R_s^3 \int_0^x \rho(y) y^2 dy. \tag{6}$$

G_N is Newton's gravitation constant. M_x is the mass enclosed in a sphere of radius $r = xR_s$. Thus, even the radial distribution of the mass density and of the pressure are fixed by the solar mass and by an evaluation of the velocity of sound. Only with respect to the distribution of $\tilde{N}_\alpha$ alpha-particles produced within the sun, there remains some freedom. Different particle distributions imply different diffusion constants, different temperature distributions and different fluxes of solar neutrinos. By the measurement of the solar neutrino flux also these uncertainties can be removed. The so-called solar neutrino problem is solved, if the SSM can be specified in such a way that the experimental values can be reproduced. Section 2 deals with the production, diffusion and distribution of the (additional) α-particles. In section 3 the various types of neutrinos are discussed. Section 4 presents the results.

2. Distribution and Diffusion of α-Particles

Using the numerical values

$$R_s = 6.9626(7) \times 10^8 \text{ m}, \quad G_N = 6.67259(85) \times 10^{-11} \text{ kg}^{-1} \text{ m}^3 \text{ s}^{-2}$$
$$M_s = 1.9889(3) \times 10^{30} \text{ kg}, \quad L_s = 3.846(5) \times 10^{26} \text{ W}$$
$$a_{tr} = 3.15569 \times 10^7 \text{ s},$$

where a_{tr} means a tropical year, and starting with a mass density distribution proposed by Stix [Stix91], one gets functions $P(x)$ and $c_s(x)$, which roughly reproduce the solar vibration frequencies (Table 1). An

Table 1. Mass density ρ, pressure P and velocity of sound v_s as functions of $x = r/R_s$

x	ρ kg m^{-3}	M_x kg	P Pa	c_s km s^{-1}
0.00	1.530 + 5	0	2.382 + 16	394.5
0.03	1.427 + 5	5.582 + 27	2.244 + 16	396.5
0.06	1.200 + 5	4.031 + 28	1.913 + 16	399.3
0.09	9.520 + 4	1.173 + 29	1.512 + 16	398.5
0.12	7.407 + 4	2.350 + 29	1.135 + 16	391.5
0.15	5.693 + 4	3.853 + 29	8.196 + 15	379.4
0.18	4.308 + 4	5.566 + 29	5.730 + 15	364.7
0.21	3.210 + 4	7.363 + 29	3.903 + 15	348.7
0.24	2.356 + 4	9.133 + 29	2.606 + 15	332.6
0.27	1.710 + 4	1.079 + 30	1.715 + 15	316.7
0.30	1.226 + 4	1.229 + 30	1.119 + 15	302.0
0.33	8.720 + 3	1.359 + 30	7.280 + 14	288.9
0.36	6.139 + 3	1.471 + 30	4.713 + 14	277.1
0.39	4.389 + 3	1.566 + 30	3.027 + 14	262.6
0.42	3.158 + 3	1.644 + 30	1.975 + 14	250.1
0.45	2.219 + 3	1.708 + 30	1.295 + 14	241.6
0.48	1.594 + 3	1.759 + 30	8.610 + 13	232.4

error is given by the number in the brackets. It refers to the last significant digitals of the preceding number

$$6.67259(85) = 6.67259 \pm 0.00085,$$

for example. Thermonuclear reactions and diffusion are concentrated to the central region. Therefore, the variables in Table 1 are restricted to $x < 0.5$. Furthermore, the notation $m \times 10^n = m + n$ is used.

One gets a final number $\tilde{N}_\alpha = 1.15 \times 10^{55}$ of α-particles which have been produced in the sun, if one assumes that within the last 4.6 $\mathrm{Ga_{tr}}$ the solar radiation power L of the sun increased from $0.71L_s$ to the present value L_s. If diffusion is neglected, then the present or final abundance $\tilde{Y}_f$ has a radial distribution

$$\tilde{Y}_0 \propto dL/dV$$

proportional to the α-particle production rate. Such distributions can be approximated by functions of the type

$$\begin{aligned} \tilde{Y}_f(x) &= Y_1 \exp(-88x^2) && (0.00 < x < 0.06) \\ \tilde{Y}_f(x) &= Y_1 \exp[100\beta(0.06^{1.6} - x^{1.6}) - 0.3168] && (0.06 < x < 0.48) \\ \tilde{Y}_f(x) &= 0, && (0.48 < x < 1.00) \end{aligned} \tag{7}$$

Table 2. Helium abundance parameter Y_1, central temperature T_c, contribution of the pp- and pep-cycles to the radiation power, mass abundance X_6 of carbon plus nitrogen, central abundances X_c and Y_c of hydrogen and helium, respectively, as a function of the model parameter β

β	Y_1	T_c MK	L_{HH} 10^{26} W	X_6 10^{-3}	X_c	Y_c
0.25	0.2485	13.99	3.755	2.733	0.478	0.505
0.30	0.2891	14.68	3.744	1.955	0.438	0.545
0.38	0.3561	15.48	3.718	1.170	0.374	0.609

where the index f refers to the final or present state. The coefficient Y_1 has to be determined from Eq. (3) for a given number $\tilde{N}_\alpha$ (Table 2). At least on the average, the value $\beta = 0.38$ corresponds to vanishing diffusion. Assuming total ionization in the radial region $x < 0.48$, neglecting the dependence of the electron pressure on the electron number density n_e, but taking into account the radiation pressure, it is possible to determine the chemical composition ($X = 0.983 - Y$, $Y = 0.256 + \tilde{Y}_f$, relative molecular mass μ) and the temperature T. The corresponding values in the center of the sun are shown in Table 2.

Using the reaction rates for the dominant thermonuclear reactions according to Fowler et al. [Fowler75, Harris83, Caughlan85], the power densities dL_{HH}/dV and $X_6^{-1} dL_{HC}/dV$ can be evaluated. Then the total power L_s determines the carbon mass abundance X_6. The numbers in Table 2 show that the carbon abundance is too low for the case $\beta = 0.38$. Furthermore, the contribution L_{HH} to the radiation power increases with decreasing central temperature. The ratio $L_{HH}/L_s = 0.971(4)$ is high. Therefore, an increase of the effective coupling constant for beta-decay and deuteron fusion would be incompatible with the SSM.

Within the radial region $x_i < x < x_{i+1}$ the number

$$(\Delta\tilde{N}_{\alpha 0})_{i,i+1} = (\tilde{N}_\alpha/L_s)4\pi R_s^3 \int_{x_i}^{x_{i+1}} (dL/dV)x^2 dx \qquad (8)$$

of α-particles has been produced during the last 4.6 Ga_{tr}. Presently the same radial interval contains the final number

$$(\Delta\tilde{N}_{\alpha f})_{i,i+1} = N_A(^4\text{He})4\pi R_s^3 \int_{x_i}^{x_{i+1}} \tilde{Y}_f(x)\rho(x)x^2 dx \qquad (9)$$

of synthesized α-particles. The difference between the numbers (8) and (9) is due to the flux of α-particles in radial direction. The gradient $\partial\tilde{n}_\alpha/\partial r$

Table 3. Diffusion constant D and diffusion length λ_D for three model parameters β

β	$(x = 0.03)$	$(x = 0.12)$	$(x = 0.21)$	$(x = 0.30)$
		$D/\text{cm}^2\,\text{s}^{-1}$		
0.25	14.8	61.70	75.58	64.07
0.30	0.9	27.97	42.63	38.47
0.38	– 10.60	– 5.99	2.61	– 1.00
		λ_D/nm		
0.25	5.04	22.43	31.74	31.35
0.30	0.30	10.15	17.94	18.84
0.38	– 3.44	– 2.17	1.10	– 0.49

and thus also the flux $\phi_\alpha(x, t)$ increases linearly with the time t

$$\phi_\alpha(x,t) = (t/4.6\,\text{Ga}_{\text{tr}})\phi_{\alpha\text{f}}(x) \quad (0 < t < 4.6\,\text{Ga}_{\text{tr}})$$
$$(\Delta\tilde{N}_{\alpha 0} - \Delta\tilde{N}_{\alpha\text{f}})_{i,i+1} = 2.3\,\text{Ga}_{\text{tr}}\,4\pi R_s^2[x_{i+1}^2\phi_{\alpha\text{f}}(x_{i+1}) - x_i^2\phi_{\alpha\text{f}}(x_i)], \tag{10}$$

where $\phi_{\alpha\text{f}}(x)$ is the present flux of α-particles at the distance $r = xR_s$ from the solar center. From Eq. (10) and the condition $\phi_{\alpha\text{f}}(0) = 0$ the present flux $\phi_{\alpha\text{f}}$ of α-particles can be determined. If this flux is due to diffusion

$$\phi_{\alpha\text{f}} = \text{D}(\partial\tilde{n}_\alpha/\partial r)_{\text{f}}, \quad \text{D} = v_0\lambda_D, \quad v_0 = (3\text{k}_\text{B}T/M_4)^{1/2}, \tag{11}$$

then the diffusion constant D and the diffusion length λ_D can be evaluated. M_4, k_B and v_0 mean the mass of a ^{4}He-atom, Boltzmann's constant and the mean thermal velocity, respectively. Table 3 shows the diffusion constant and length for various model parameters β. It can be seen that small diffusion cannot be achieved in a self-consistent way: The negative constants for $\beta = 0.38$ cannot be justified physically. Even in the case $\beta = 0.30$ the variation of the diffusion length is too strong. Only in the case $\beta = 0.25$ the length λ_D is restricted to a narrow and physically meaningful range $10 < n_e\lambda_D < 460$ for $\beta = 0.30$, $200 < n_e\lambda_D < 810$ for $\beta = 0.25$. Both, the carbon abundance and the diffusion parameters suggest a model parameter $\beta = 0.25$.

3. Neutrino Flux

Six processes contribute to the solar neutrino production. N_i is the total number of neutrinos coming from the i-th process

$$\begin{aligned}
\dot{N}_1 &: \quad {}^1\mathrm{H} + {}^1\mathrm{H} \to {}^2\mathrm{H} + \mathrm{e}^- + \mathrm{e}^+ + \nu \\
\dot{N}_2 &: \quad {}^1\mathrm{H} + {}^1\mathrm{H} \to {}^2\mathrm{H} + \nu \\
\dot{N}_7 &: \quad {}^7\mathrm{Be} \to {}^7\mathrm{Li} + \nu \\
\dot{N}_8 &: \quad {}^8\mathrm{B} \to {}^8\mathrm{Be} + \mathrm{e}^- + \mathrm{e}^+ + \nu \\
\dot{N}_{13} &: \quad {}^{13}\mathrm{N} \to {}^{13}\mathrm{C} + \mathrm{e}^- + \mathrm{e}^+ + \nu \\
\dot{N}_{15} &: \quad {}^{15}\mathrm{O} \to {}^{15}\mathrm{N} + \mathrm{e}^- + \mathrm{e}^+ + \nu
\end{aligned}$$

The total production rate $\dot{N}_\nu$ of neutrinos is given by the relations

$$\begin{aligned}
\dot{N}_\nu &= \dot{N}_{\mathrm{HH}} + \dot{N}_{\mathrm{HC}} = 1.8341(4) \times 10^{38}\ \mathrm{s}^{-1} \\
\dot{N}_{\mathrm{HH}} &= \dot{N}_1 + \dot{N}_2 + \dot{N}_7 + \dot{N}_8 = 2.3813 \times 10^{11}\ \mathrm{s}^{-1}\ L_{\mathrm{HH}}/\mathrm{W} \qquad (12) \\
\dot{N}_{\mathrm{HC}} &= \dot{N}_{13} + \dot{N}_{15} = 2.4925 \times 10^{11}\ \mathrm{s}^{-1}\ L_{\mathrm{HC}}/\mathrm{W}
\end{aligned}$$

The variation of the total rate with the model parameter β is very small. It is indicated by the number in the brackets. To each process there corresponds a neutrino flux ϕ_i on the earth

$$\phi_i = (\dot{N}_i/4\pi r_{\mathrm{es}}^2), \quad r_{\mathrm{es}} = 1.4959787 \times 10^{11}\ \mathrm{m}, \qquad (13)$$

where r_{es} is the mean distance between the centers of the earth and of the sun. The total neutrino flux is given numerically by $\phi_\nu = 6.52 \times 10^{14}\ \mathrm{m}^{-2}\ \mathrm{s}^{-1}$. The special fluxes ϕ_i are shown in Table 4.

The comparison with experiment is encouraging. The Kamiokande collaboration [Kamionkowski94, Suzuki95] observes the Cerenkov radiation emitted by electrons, which have been scattered by high energy neutrinos. A neutrino flux

$$\phi_{\mathrm{K}} = 2.73(51) \times 10^{10}\ \mathrm{m}^{-2}\ \mathrm{s}^{-1} \qquad (14)$$

has been measured. Mainly the boron neutrinos contribute to this flux: $\phi_8 < \phi_{\mathrm{K}}$. This inequality demands the condition: $\beta < 0.3$. Especially for $\beta = 0.25$, there is a flux $\phi_8 = 1.66 \times 10^{10}\ \mathrm{m}^{-2}\ \mathrm{s}^{-1}$ of neutrinos due to the

Table 4. Neutrino flux ϕ_i in units of $\mathrm{m}^{-2}\ \mathrm{s}^{-1}$ for three model parameters β

	$(\beta = 0.25)$	$(\beta = 0.30)$	$(\beta = 0.38)$
ϕ_1	6.096 + 14	6.024 + 14	5.873 + 14
ϕ_2	1.449 + 12	1.431 + 12	1.396 + 12
ϕ_7	2.481 + 13	3.030 + 13	4.087 + 13
ϕ_8	9.435 + 09	1.702 + 10	4.820 + 10
ϕ_{13}	8.085 + 12	8.998 + 12	1.133 + 13
ϕ_{15}	8.085 + 12	8.998 + 12	1.133 + 13
ϕ_ν	6.520 + 14	6.521 + 14	6.523 + 14

Table 5. Reaction rate $\sigma\phi$ for the production of radioactive ^{37}Ar and ^{71}Ge

β	$\sigma\phi(^{37}\mathrm{Cl})$ [snu]	$\sigma\phi(^{71}\mathrm{Ga})$ [snu]
0.25	2.50	109
0.30	3.53	116
0.38	7.33	135
exper.:	2.55(35)	70(8)

^{8}B-decay, and a flux $1.07(51) \times 10^{10}\,\mathrm{m}^{-2}\,\mathrm{s}^{-1}$ comes from the other processes altogether. Now the processes

$$\nu + {}^{37}\mathrm{Cl} \to {}^{37}\mathrm{Ar}^+ + \mathrm{e}^-, \quad \nu + {}^{71}\mathrm{Ga} \to {}^{71}\mathrm{Ge}^+ + \mathrm{e}^-$$

shall be considered. On the average over the energy spectrum of each neutrino flux ϕ_i, the cross sections σ_i have been calculated [Bahcall79, Hampel86]. Here the compilation of Stix [Stix91] is used. Table 5 shows the corresponding reaction rates

$$\sigma\phi = \sum_i \sigma_i \phi_i \quad (i = 1, 2, 7, 8, 13, 15)$$

in solar neutrino units (1 snu $= 10^{-36}\,\mathrm{s}^{-1}$ per target atom). The experimental reaction rate found by the chlorine collaboration [Cleveland95] is only compatible with the parameter $\beta = 0.25$. The experimental reaction rate found by the Gallex collaboration [Hampel96] always is smaller than the values expected theoretically. But for $\beta = 0.25$ the ratio experiment versus theory reaches the highest value

$$\sigma\phi(\mathrm{Ga,\ exper.})/\sigma\phi(\mathrm{Ga},\ \beta = 0.25) = (64 \pm 8)\,\%.$$

Here a still higher ratio can be expected for the future.

4. Conclusion

The diffusion of the α-particles produced within the sun, the mass abundance of carbon, the Kamiokande and the chlorine experiments suggest a parameter $\beta = 0.25$. At present the gallium experiment reaches 2/3 of the theoretical value for $\beta = 0.25$. Therefore, the solar neutrinos do not appear as a problem, but as a suitable tool to determine the diffusion parameters in the sun. The favoured parameter $\beta = 0.25$ means a central temperature $T_\mathrm{c} = 14.0\,\mathrm{MK}$ and a carbon mass abundance $X_6 = 2.7 \times 10^{-3}$. The CNO-cycle contributes 2.4 % to the radiation power ($L_\mathrm{HC} =$

$0.0237L_s$). The total radiation power includes the neutrino emission

$$L_{\text{tot}} = 1.021L_s = 3.927 \times 10^{26}\,\text{W}.$$

The contribution of the radiation power P_γ is small

$$P_\gamma = 5(1) \times 10^{-4}\,P.$$

The diffusion constant D for α-particles varies in the range of $(20\ldots 80)\,\text{cm}^2\,\text{s}^{-1}$, and in the case of radiation equilibrium the opacity κ should increase with increasing distance from the center

$$\kappa = (0.057\ldots 0.462)\,\text{m}^2\,\text{kg}^{-1} \quad \text{for} \quad x = 0.06\ldots 0.36.$$

Smaller values of β mean a lower central temperature and a smaller flux of high energy neutrinos. Especially for boron neutrinos one gets

$$\phi_8 = 4.0(4) \times 10^{-9}\,\text{m}^{-2}\,\text{s}^{-1}\,(T_c/\text{MK})^{16} \quad \text{for} \quad T_c = 15(1)\,\text{MK}.$$

A decrease in the central temperature from $T_c = 15.5$ MK to 14.0 MK is sufficient for a decrease of the theoretical reaction rate in the chlorine process from 7.3 snu to the experimental value of 2.5 snu.

If the velocity of sound, particle diffusion and the flux of solar neutrinos are included in the boundary conditions for the internal solar observables, then the neutrino production does not contradict the SSM, but this solar model becomes specified with respect to the local chemical composition and with respect to the radial distribution of the temperature. Although some simplifications have been made, it should be clear that it is not necessary to abandon the SSM. It rather can be specified in a satisfactory way.

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