The Invariant Measure
for the Two-Dimensional Parry-Daniels Map

By

Fritz Schweiger

durch das k. M. I. Fritz Schweiger)

Abstract

The Parry-Daniels map $T$ has an exceptional set $\Gamma$ which can be seen as a strange
attractor for $T$. The density of the invariant measure is given. Some remarks on the
exceptional set for the mixture of the Selmer algorithm and the fully subtractive
algorithm are added.

Key words: Ergodic theory, invariant measures.

Let $x = (x_0, x_1, x_2) \in (\mathbb{R}_+^3)$ and let $\pi$ be a permutation of the indices
such that $x_{\pi 0} \leq x_{\pi 1} \leq x_{\pi 2}$. The Poincaré map $P$ is defined as

$$P(x_0, x_1, x_2) = (x_{\pi 0}, x_{\pi 1} - x_{\pi 0}, x_{\pi 2} - x_{\pi 1}).$$

We introduce

$$\Sigma^2 = \{x \in (\mathbb{R}_+^3) : x_0 + x_1 + x_2 = 1\}.$$

Then the Parry-Daniels map $T : \Sigma^2 \to \Sigma^2$ is defined as

$$T(x_0, x_1, x_2) = \left(\frac{x_{\pi 0}}{x_{\pi 2}}, \frac{x_{\pi 1} - x_{\pi 0}}{x_{\pi 2}}, \frac{x_{\pi 2} - x_{\pi 1}}{x_{\pi 2}}\right),$$

$$\pi \in \{\varepsilon, (01), (02), (12), (012), (021)\}.$$
We introduce the notation
\[ x^{(k)} = (x^{(k)}_0, x^{(k)}_1, x^{(k)}_2) := P^k x. \]
We define
\[ \sigma(x) := \sum_{k \geq 0} \max(x^{(k)}_0, x^{(k)}_1). \]

The following result could be proved (SCHWEIGER [2], NOGUEIRA [1]).

Let
\[ \Gamma := \bigcap_{s=0}^{\infty} \bigcup_{\pi_1, \ldots, \pi_s \in \{\varepsilon, 01\}} B(\pi_1, \ldots, \pi_s), \]
then \( \Gamma = \{ x \in \Sigma^2 : \sigma(x) \leq x_2 \} \) and \( \lambda(\Gamma) > 0 \). Since \( T \) is ergodic with respect to Lebesgue measure, we obtain
\[ \Sigma^2 = \bigcup_{j=0}^{\infty} T^{-j} \Gamma. \]

Note that \( \sigma(x) \) is convergent for almost all directions \( \theta = x_0/x_1 \) or \( \theta = x_1/x_0, \ 0 \leq \theta \leq 1. \)

Since on \( \Sigma^2 \) the relation \( x_2 = 1 - x_0 - x_1 \) holds, we restrict our attention to the first coordinates, i.e. to the domain \( \{(x_0, x_1) : 0 \leq x_0, \ 0 \leq x_1, 0 \leq x_0 + x_1 \leq 1\} \).

**Theorem.** The function
\[ h(x_0, x_1) = \frac{1}{x_0(x_0 + x_1)(1 - x_0 - x_1 - \sigma(x_0, x_1))} \]
is an invariant density for \( T \) restricted to \( \Gamma \).

**Proof.** The map \( T \) restricted to \( \Gamma \) has only two inverse branches
\[ V(\varepsilon)(x_0, x_1) = \left( \frac{x_0}{1 + 2x_0 + x_1}, \frac{x_0 + x_1}{1 + 2x_0 + x_1} \right), \]
\[ V(01)(x_0, x_1) = \left( \frac{x_0 + x_1}{1 + 2x_0 + x_1}, \frac{x_0}{1 + 2x_0 + x_1} \right). \]

Then
\[ h(V_0(\varepsilon)(x_0, x_1))\omega(\varepsilon, x_0, x_1) + h(V(01)(x_0, x_1))\omega(01; x_0, x_1) \]
\[ = \frac{1}{x_0(2x_0 + x_1) \left( 1 - (1 + 2x_0 + x_1)\sigma\left( \frac{x_0}{1 + 2x_0 + x_1}, \frac{x_0 + x_1}{1 + 2x_0 + x_1} \right) \right)} \]
We note the following properties of the function \( \sigma \):

\[
\sigma(\lambda y_0, \lambda y_1) = \lambda \sigma(y_0, y_1), \\
\sigma(y_0, y_1) = \sigma(y_1, y_0), \\
\sigma(x_0, x_0 + x_1) = x_0 + x_1 + \sigma(x_0, x_1).
\]

Therefore

\[
(1 + 2x_0 + x_1) \sigma \left( \frac{x_0}{1 + 2x_0 + x_1}, \frac{x_0 + x_1}{1 + 2x_0 + x_1} \right) = x_0 + x_1 + \sigma(x_0, x_1).
\]

Hence

\[
h(V(\varepsilon)(x_0, x_1))\omega(\varepsilon; x_0, x_1) + h(V(01)(X_0, x_1))\omega(01; x_0, x_1) = h(x_0, x_1).
\]

**Remark 1.** The set \( \Gamma \) can be described as consisting of all needles emanating from \((0,0)\) which are given by the equations

\[
x_0 = \lambda, \quad x_1 = \lambda \theta, \quad 0 \leq \lambda \leq \frac{1}{1 + \theta + S(\theta)},
\]

or

\[
x_0 = \lambda \theta, \quad x_1 = \lambda, \quad 0 \leq \lambda \leq \frac{1}{1 + \theta + S(\theta)},
\]

\[
S(\theta) = \sigma(\theta, 1) = \sigma(1, \theta), \quad 0 \leq \theta \leq 1.
\]

Therefore the equation

\[
x_0 + x_1 + \sigma(x_0, x_1) = 1
\]

can be viewed as referring to the boundary of \( \Gamma \) in some sense (the other parts of the boundary are given by \( x_0 = 0 \) and \( x_1 = 1 \)).

**Remark 2.** This remark concerns the paper SCHWEIGER [3]. In this paper the Selmer algorithm \( S \) and the Fully Subtractive algorithm \( T \) were considered. The following theorem was proved:

**Theorem.** Let \( \Gamma = (x_1, x_2) \in B^2: (S \circ T)^j x \in E, \ j \geq 0. \) Then \( \lambda(\Gamma) > 0. \)

The proof given was a modification of SCHWEIGER [2]. The essential idea is to show that

\[
\frac{q_n}{A_n} \geq \gamma > 0, \quad \gamma = \gamma(u) \quad \text{a.e.}
\]
However in contrast to the Parry-Daniels map it is easy to show that there is a constant $\gamma > 0$ such that for all $u$

$$\frac{q_n}{A_n} \geq \gamma > 0.$$  

From

$$a_{n+1} \leq \frac{q_{n+1}}{q_n}$$

one sees by induction that

$$q_n \leq \left( 2 + \frac{1}{q_1} + \cdots + \frac{1}{q_{n-1}} \right) A_n$$

holds. This implies that the set $\Gamma$ contains a triangle. Therefore the set $\Gamma$ is less “exceptional” as explained in Remark 2. In fact, $\Gamma$ contains the triangle with the vertices $(0,0), \left( \frac{1}{2}, 0 \right), \left( \frac{1}{3}, \frac{1}{3} \right)$. But it is easy to see that $\Gamma$ contains at least countably many segments which start at $(0,0)$ but go beyond the line $2x_1 + x_2 = 1$.

The restriction of $S \circ T$ on $\Gamma$ has the $\sigma$-finite invariant measure with density

$$h(x_1, x_2) = \frac{1}{x_1x_2(1 - 2x_1 - x_2)}.$$

**Acknowledgement**

This paper was inspired by discussions on the dynamics of $T$ on $\Gamma$ during the Workshop on Dynamical Systems and Number Theory in Strobl (July 2007).

**References**


**Author’s address:** Prof. Dr. Fritz Schweiger, Department of Mathematics, University of Salzburg, Hellbrunner Strasse 34, 5020 Salzburg, Austria. E-Mail: fritz.schweiger@sbg.ac.at.