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On Some Problems Concerning Expansions by Non Integer Bases

By

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Consider a real number $1 < q < 2$ and take any expansions of 1 of the form

$$1 = \sum_{i=1}^{\infty} q^{-n_i} \tag{1}$$

where n_i is a strictly increasing sequence of natural numbers. Define further

$$0 = y_0 < y_1 < y_2 < \dots$$

as the increasing sequence of real numbers y which have at least one representation of form

$$y = \varepsilon_0 + \varepsilon_1 q + \varepsilon_2 q^2 + \dots + \varepsilon_n q^n \tag{2}$$

where $\varepsilon_1, \dots, \varepsilon_n \in \{0, 1\}$. Such an ordering clearly exists since in bounded intervals there are only finitely many y satisfying (2). If q varies, the coefficients of $y_n = y_n(q)$ in (2) may vary as well. For fixed q, define

$$\ell(q) = \liminf_{n\to\infty} (y_{n+1} - y_n), \quad L(q) = \limsup_{n\to\infty} (y_{n+1} - y_n). \tag{3}$$

Clearly $0 \leq \ell(q)$ for all q. More is given in [2], namely:

Theorem A [2].
a) $L(q) \leq 1 \ \forall q$.
b) $L(q) = 1$ *if* $q \geq A := (1 + \sqrt{5}/2)$, *i.e.* $y_{n+1} - y_n = 1$ *for infinitely many n.*
c) $L(q) = 0$ (*i.e.* $y_{n+1} - y_n \to 0$) *implies that there exists an expansion* (1) *satisfying*

$$\sup\,(n_{i+1} - n_i) = \infty. \tag{4}$$

(*This means that there are arbitrarily long sequences of consecutive* 0 *as digits in the expansions of* 1).

The essential point of the proof of c) is a recursive construction of the digits of (1) *based on the fact that there exists a large m, for which the quantity*

$$q^m \cdot \left(1 - \sum_{i=1}^{i_k} q^{-n_i}\right)$$

can be approximated from below by some y_n *with an error as small as we please. Hence the same proof implies.*

Theorem 1. *Let* $1 < q < 2$ *and suppose that for every* $x > 0$ *and for every* $\varepsilon > 0$ *there exist m,* $n \in N$ *with*

$$q^m \cdot x - \varepsilon < y_n < q^m x. \tag{5}$$

Then there exists an expansion (1) *of* 1 *satisfying*

$\sup\,(n_{i+1} - n_i) = \infty$.

Remark. If q is not algebraic, all numbers of the form (2) are different. Since there are 2^n numbers of the form (2) and the largest number $1 + q + \ldots + q^n = (q^{n+1} - 1)/(q-1)$ is much less than 2^n for large n we get that for transcendental q, $\ell(q) = 0$. It is clearly enough that q is not a zero of a polynomical with coefficients 0 and ± 1. Hence

$$\ell(q) = 0 \quad \text{for} \quad 1 < q < 2 \tag{6}$$

with denumerable exceptions.

In [6], [1] much more is proved. We introduce the notion of Pisot-numbers (PV–numbers in [4], [5]) as algebraic integers $q > 1$ whose conjugates are complex numbers with modulus < 1. It is known that the least Pisot number is the positive root q_1 of the polynomial

$$x^3 = x + 1$$

and its approximate value is $q_1 \approx 1{,}324717957$. There holds

Theorem B ([6], [1]).
a) If $q\in(1,2)$ *is a Pisot-number then* $\ell(q)>0$
b) If $q\in(1,2)$ *is the root of* $x^{r+1}-\Sigma_{k=0}^{r}x^{k}=0$ *then* $\ell(q)=1/q$. *In particular, for* $q=A$ *we have* $\ell(q)=1/A$.

Here the following question arises:

Problem 1. Find the converse of Theorem A. *c*)

Remark. Of course, sup $(n_{i+1}-n_i)=\infty$ does not imply $L(q)=0$ since for a.a.$q: 1<q<2$ we have sup $(n_{i+1}-n_i)=\infty$ (see [8]) and for $A\geq q$, $L(q)=1$.

Problem 2. Does the coverse of Theorem 1 hold?

The implication holds for $x=1$ since if sup $(n_{i+1}-n_i)=\infty$ then

$$\forall k \quad \exists m: 0<1-\sum_{i=1}^{m}\varepsilon_i q^{-i}<q^{-m-k}$$

with some $\varepsilon_i\in\{0,1\}$. This means that we have

$$q^{m}-\frac{1}{q^{k}}<\sum_{i=1}^{m}\varepsilon_i q^{m-i}<q^{m},$$

i.e. (5) holds indeed with $x=1$. Then clearly

$$q^{m}(1+\delta_1 q+\ldots+\delta_n q^{n})-\frac{1}{q^{k}}<\sum_{i=1}^{m}\varepsilon_i q^{m-i}+\sum_{j=1}^{m}\delta_j q^{m+j}$$
$$<q^{m}(1+\delta_1 q+\ldots+\delta_n q^{n})$$

hence $x=1+\delta_1 q+\ldots+\delta_n q^{n}$ also satisfies (5) if $\delta_i\in\{0,1\}$ is arbitrary. If (5) holds for x then it holds also for $x/q^{1}, x/q^{2}$ ect. We thus get (5) for every finite sum

$$\sum\delta_j q^{j}$$

$\delta_j\in\{0,1\}$ where the j are integer (positive or negative) indices. The set of such sums is dense in $(0,\infty)$ by Theorem A a). On the other hand the set of "good" x,

$$X=\{x>0:\forall\varepsilon>0 \quad \exists m,n\in N \quad \text{with} \quad q^{m}x-\varepsilon<y_n<q^{m}x\}$$

is a G_δ-set, i.e. the intersection of countable many open sets G_k where

$$G_k=\bigcup_{m,n}\left\{x>0:q^{m}x-\frac{1}{k}<y_n<q^{m}x\right\}.$$

Hence

$$X=\bigcap_{k=1}^{\infty} G_k$$

is a dense G_δ-set and thus $(0, \infty)\backslash X$ is a set of I. category.

Problem 2. (reformulated). If there exists an expansion (1) with $\sup(n_{i+1}-n_i)=\infty$ then $X=(0, \infty)$.

Problem 3. It is ture that

$$X=(0, \infty) \Leftrightarrow \ell(q)=0\,?$$

Problem 3′. There exists an expansion (1) with $\sup(n_{i+1}-n_i)=\infty \Leftrightarrow \ell(q)=0$.

Remark. This last conjecuture holds a.c. since for a.a. $q\in(1, 2)$ both parts of the equivalence are true. The only if part of Problem 3′ is trivial; we have seen that if $\sup(n_{i+1}-n_i)=\infty$ then q^m can be well approximated from below by some y_n, hence $\ell(q)=0$ holds indeed. To see the converse it would be enough to show the only if part of Problem 3, i.e. that $\ell(q)=0$ implies $X=(0, \infty)$.

Problem 4.

$\ell(q)>0 \Leftrightarrow q$ is Pisot.

The if part is already mentioned in Theorem B. Concerning the only if part the following result of Y. Bugeaud [3] is to be mentioned:

Theorem C [3] *$q\in(1, 2)$ Pisot $\Leftrightarrow$ $\liminf_{n\to\infty}(y_{n+1}^{(k)}-y_n^{(k)})>0$ for all natural integers k where*

$$0=y_0^{(k)}<y_1^{(k)}<\dots$$

is the increasing rearrangement of all y having at least one representation

$$y=\varepsilon_1 q+\dots+\varepsilon_n q^n;\quad \varepsilon_1,\dots,\varepsilon_n\in\{0, 1,\dots, k\}.$$

To see the basic ideas used here (borrowed from the theory of automata) we need the following notions.

Let $c\in N$, $C=\{-c,\dots,0,\dots,c\}$ and

$$Z(q, c)=\left\{s=(s_n)_{n\geq 0}\in C^N: \sum_{n\geq 0} s_n q^{-n}=0\right\},$$

formed by the infinite words in C^N *which correspond to* 0 *in the basis* q. *Let further be*

$$C^* = \bigcup_{n=1}^{\infty} C^n$$

the set of finite words over the alphabet C *and*

$$LF(Z(q, c)) = \{w \in C^* : \exists s \in C^N \quad \text{with} \quad ws \in Z(q, c)\}$$

be the set of left factors (*i.e. the 'beginnings'*) *of the words which correspond to* 0 *in the basis* q. *To every element* $(s_0, \ldots, s_n) \in LF(Z(q, c))$ *we associate a polynomial* $F(x) = s_0 x^n + \ldots + s_n$. *Clearly* $F(q)$ *is the remainder of the Euclidean division of* $F(x)$ *by* $x - q$. *The values* $F(q)$ *are bound for fixed* q *since there are* $s_{n+1}, s_{n+2}, \ldots$ with $\Sigma_0^\infty s_i q^i = 0$, *hence*

$$|F(q)| = \left| -\sum_{n+1}^{\infty} s_i q^{n-i} \right| \leq c \sum_{n+1}^{\infty} q^{n-i} = \frac{c}{q-1}.$$

In the theory of automata the following result it is known:

Theorem D ([9], [10]). $q \in (1, 2)$ *is Pisot* $\Leftrightarrow$ *For every* $c \in N$ *the set of the remainders* $F(q)$ *is finite where* F *runs over all polynomials associated to* $F(Z(q, c))$.

In fact, in [9] *and* [10] *it is proved that both statements are equivalent to the statement that* $Z(q, c)$ *is recognizable by a finite automaton. This notion is defined as follows. A finite automaton*

$$\mathscr{A} = (C, Q, I, T)$$

over the alphabet C *is a* (*not necessarily complete*) *directed graph whose edges are labeled by letters of* C, Q *is the* (*finite*) *set of vertices called states,* $I \subset Q$ *is a subset of the so-called initial states,* $T \subset Q$ *consists of the terminal states. Denote again by* C^* *the set of all finite words in* C. *A word* $w \in C^*$ *is recognized by* $\mathscr{A}$ *if it is the label sequence of a path in* $\mathscr{A}$ *starting from* I *and arriving in* T. *A subset* E *of* C^* *is called recognizable if there exists a finite automaton* $\mathscr{A}$ *over the alphabet* C *such that* E *is the set of all words recognized by* $\mathscr{A}$.

Remark. The condition of Theorem D can be reformulated as follows: $-s_0 = \Sigma_1^\infty s_i q^{-i}$ implies the numbers $x_k = \Sigma_{i \geqslant k} s_i q^{k-i}$ give only finitely many different values for all $k \in N$ and for all expansions $-s_0 = \Sigma_1^\infty s_i q^{-i}$, $|s_i| \leq c$. Almost the same has been proved earlier in [11] (with $s_0 = -1$, $s_i \in \{0, 1\}$ for $i \geq 1$) for Pisot numbers. The converse implication is not investigated in [11], therefore we pose it as a problem:

Problem 5. $q \in (1, 2)$ is Pisot $\Leftrightarrow$ the set of all $x_k = \varepsilon_k + (\varepsilon_{k+1}/q) + (\varepsilon_{k+2}/q^2) + \ldots$, $k \in N$ for every expansion $1 = \Sigma_{i=1}^\infty \varepsilon_i q^{-i}$ is finite.

Remark. If Problem 5 is true it almost solves also Problem 4. Indeed, if q is not Pisot then there are infinitely many different values x_k. Since

$$0 \le x_k \le 1 + \frac{1}{q} + \frac{1}{q^2} + \ldots = \frac{1}{1 - \frac{1}{q}}$$

is bounded, then for every $\varepsilon > 0$ there exist integers k, ℓ with

$$0 < |x_k - \tilde{x}_\ell| < \varepsilon$$

where $\tilde{x}_\ell$ is the ℓ-th truncation of another expansion of 1. On the other hand

$$x_k = q^k - \sum_{i=1}^{k} \varepsilon_i q^{k-i}, \quad \varepsilon_i \in \{0, 1\}.$$

Hence in $x_k - \tilde{x}_\ell$ all coefficients are in $\{-1, 0, 1\}$ and at most one coefficient can be ± 2. Hence $\liminf_{n\to\infty} (y_{n+1}^{(2)} - y_n^{(2)}) = 0$ follows and if there were cases where the 'bad' coefficient ± 2 does not occur then $\ell(q) = 0$ would also follow.

It is interesting to investigate the connection between $\ell(q)$ and $L(q)$. We know that $0 \le \ell(q) \le L(q) \le 1$, that $\ell(q) = 0$ with countably many exceptions and that $L(q) = 1$ for $q \ge A$. We have further:

Theorem E [1].

$$1 < q < \sqrt{2}, \ell(q^2) = 0 \to L(q) = 0.$$

The basic idea for the proof comes from the following:

Lemma [1]. *If $\ell(q) = 0$, $q \in (1, 2)$ then for every $\varepsilon > 0$ and $D > 0$ there exists a finite subsequence*

$$w_0 < w_1 < \ldots < w_m$$

of the sequence y_k such that

$$w_m - w_0 > D, \quad w_i - w_{i-1} < \varepsilon \quad (i = 1, \ldots, m).$$

By this Lemma, we can prove Theorem E by using the odd powers of q for the rough approximation, then the even powers for the fine approximation. Namely, if $\ell(q^2) = 0$, we build a sequence $w_0 < \ldots < w_m$ using the powers q^2 with $D > q$. Now if $x > w_0 + q$, there exists a sum

$$y_{2k+1} = \varepsilon_1 q + \varepsilon_3 q^3 + \ldots + \varepsilon_{2n+1} q^{2n+1}$$

with the property

$$w_0 + y_{2k+1} \leq x \leq w_0 + y_{2k+1} + q \leq w_m + y_{2k+1}$$

since $L(q^2) \leq 1$. B*y the Lemma, there exists a* j: $0 \leq j \leq m$ *with*

$$|y_\ell - x| = |w_j + y_{2k+1} - x| < \varepsilon$$

if $x > w_0 + q$, *hence* $L(q) = 0$ *indeed.*

In the above proof we used only finitely many digits of even indices (*for fixed* $\varepsilon > 0$) *and we have some freedom in choosing these digits. This freedom might be enough to prove*

Problem 6.

$$1 < q < A, \ell(q) = 0 \rightarrow L(q) = 0$$

As we have mentioned, for $q > A$ this can not be true since $L(q) = 1$ and $\ell(q) = 0$ a.e.q. We know further that $\ell(q) = 0$ with countable many exceptions; the smallest known (for us) values q with $\ell(q) > 0$ is the least Pisot-number q_1.

Problem 7. If

$$1 < q < q_1 \quad \text{does then follow} \quad \ell(q) = 0?$$

If this is true, it implies that $L(q) = 0$ for $1 < q < \sqrt{q_1}$. This would answer a problem from [1] stating that for the q near 1 we have $L(q) = 0$. The largest interval possible where $L(q) = 0$ is $(1, q_1)$ where q_1 is the smallest Pisot number. Since $q_1 < \sqrt{2}$, the problem of [1] stating that $L(q) \equiv 0$ for $q < \sqrt{2}$, fails.

We can attack Problem 7, e.g. by "principle of boxes" : in the interval

$$(0, 1 + q + \ldots + q^{n-1});$$

there are 2^n formally different sums $\varepsilon_0 + \varepsilon_1 q + \ldots + \varepsilon_{n-1} q^{n-1}$, so in a segment of length $0(q^n)$, 2^n values are distributed. If $\ell(q) > 0$ then there are values $y = y_k$ having $\geq (q-1)(2/q)^n$ different representations. Substracting any two representations we get a polynomial with coefficients 0 or ± 1 having the value 0 for $x = q$. Such a polynomial can be divided by the minimal polynomial of q. So the following question arises.

Problem 8. Given any polynomial $p_0(x) = a_0 x^k + \ldots + a_k$ with integer coefficients and with $a_0 = 1$, determine the number of polynomials $p(x) = \varepsilon_0 x^n + \ldots + \varepsilon_n$, $\varepsilon_i \in \{-1, 0, 1\}$ satisfying $p_0 | p$. To have the growth rate of this numbers would be sufficient.

This growth clearly depends on p_0. We know that if q approaches 1, the degree of p_0 tends to infinity. We guess that it happens rarely that $p_0 | p$. If the number of these p is $\leq (1+\delta)^n$ for large n then

$$(1+\delta)^n \geq (q-1)\left(\frac{2}{q}\right)^n - 1$$

which is impossible if $\delta \leq (2/q) - 1$.

Remark. Recently, A. Joó [11] investigated in detail results known in the theory of finite automata from the point of view of Number Theory.

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