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A Spacefilling Trefoil Knot

By

P. Schmitt

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Summary

This note describes a (polyhedral) prototile which is topologically equivalent to a trefoil knot and admits a (monohedral) tiling of 3-space.

1. Introduction

This note was initiated by a lecture of Schulte [6] during which he suggested to search for a spacefilling knotted polyhedron. In the sequel Danzer constructed a 4-hedral tiling by knots. It is obtained by tiling 3-space with linked congruent tori each of which is dissected into four topologically equivalent (but not congruent) knots.

The construction below shows that it is possible to obtain a monohedral tiling, i.e., to restrict oneself to a single prototile:

There is a polyhedral spacefiller which is topologically equivalent to a trefoil (or cloverleaf) knot.

2. The Knotted Spacefiller

Probably the best way to describe the prototile is to follow its construction step by step.

Step 1

The construction starts with a tiling of the plane by regular hexagons, and the observation that its hexagons can be coloured using three colours (i.e., there is a partition of the family of tiles into three classes) in such a way that the resulting three classes of tiles are pairwise congruent (see Fig. 1.)

Step 2

Two plane layers (breadth *d*) equipped with this tiling are joined by hexagonal prisms (height 2*d*) – every hexagon of one class (say: the black hexagons) is joined with a hexagon of the corresponding class on the other plane layer. In both layers the hexagons of one class (say: the white hexagons) are removed. Since the set of removed hexagons is congruent to the set of hexagons equipped with prisms it is possible to link this pair of planes with two congruent copies by putting the prisms through the holes. Therefore it is possible to fill space by copies of this linked pair of planes. (See Fig. 2: In order to make the structure more clearly visible, the breadth of the layers, and the diameter of the prisms, has been slightly reduced. The empty space caused by this is, of course, not present in an actual tiling.)

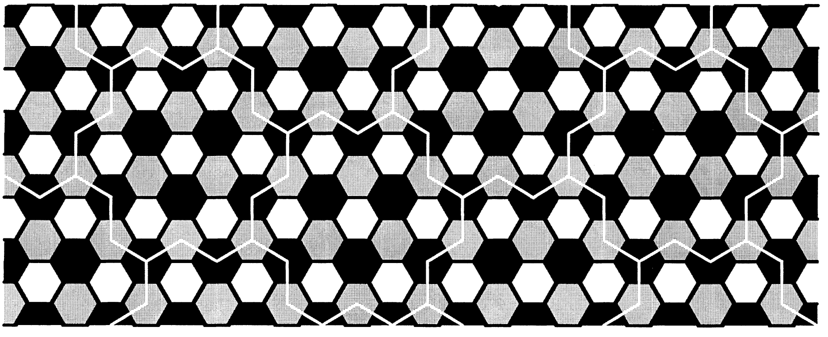

Fig. 1. 3-coloured tiling by hexagons, with composite tiles

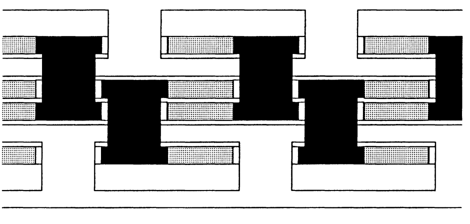

Fig. 2. Linked pairs of planes (cross section)

Step 3

Each pair of planes can be dissected into congruent – essentially hexagonal – pieces as indicated in Fig. 1. The resulting polyhedron consists of two disks (each of which has seven holes) which are joined by twelve prisms.

Step 4

Each of the pieces obtained in the previous step can be dissected into two (directly) congruent pieces which have threefold central symmetry and are topologically (or rather: homotopically) equivalent to the trefoil knot. This dissection is indicated in Fig. 3. Both the top and the bottom disk are

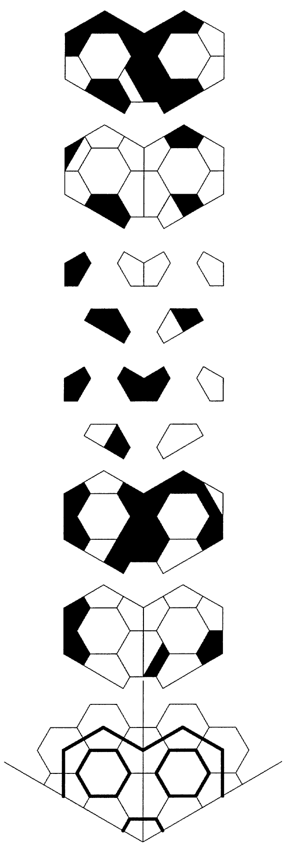

Fig. 3. Composition of the knotted tile

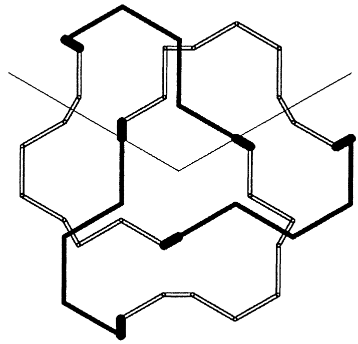

Fig. 4. The structure of the knot

partitioned into two layers (each of breadth $d/2$), and the joining prisms are partitioned into two parts (each of height d). The figure shows a third of the knot (the other two thirds are congruent to it), and (in each layer) in black those parts that belong to the polyhedron. The parts left white belong to the second copy of the knot. (The figure shows – in their natural order – the two top layers, the two parts of the prisms, and the two bottom layers, and a sketch which indicates the position of the depicted region with respect to the tiling of Fig. 1). The resulting knot has essentially the shape shown in Fig. 4. (Parts of the knot corresponding to the top layers are shown with a bold line, parts corresponding to bottom layers are shown with a double line, and spots where the knot passes (through a prism) from top to bottom are indicated by short thick black strokes. 'Dead ends', i.e., parts of the polyhedron which are only used to prevent that the complementary knotted polyhedron touches itself, or which merely serve to fill up space, are neglected. That part of the knot shown in Fig. 3 is also indicated.)

3. Remarks

Similar methods can be used to construct other knotted spacefillers. For instance, there is a monohedral tiling by a trefoil knot which exhibits more symmetries than the tiling described above (see [4]). Moreover, any arbitrary knot may serve as a model for the spacefiller to be constructed. (See the subsequent note [5].)

Other constructions of spacefilling knots are given independently by Kuperberg [2] (also stimulated by Schulte) and Adams [1], both including knotted polyhedra which tile 3-space periodically, without interlocking.

Furthermore, another example of a spacefilling trefoil knot was constructed earlier by McMullen [3].

Recently Zaks [7] gave a simplified version based on Kuperberg's approach.

References

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[6] Schulte, E.: Spacefillers of higher genus. Lecture, Austrian conference on discrete geometry. May 2–8, 1993. Neuhofen/Ybbs, Austria.
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Author's address: Dr. Peter Schmitt, Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria.