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The Exponent of Convergence for Brun's Algorithm in two Dimensions

By

B. R. Schratzberger

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durch das k. M. Fritz Schweiger)

Abstract

We show that for the two-dimensional multiplicative Brun's algorithm, the exponent of convergence is $1 + d$, i.e. there is a $d > 0$ such that for almost all $x = (x_1, x_2)$, $\left| x_i - \frac{p_i^{(t)}}{q^{(t)}} \right| \leq \frac{1}{\left(q^{(t)}\right)^{1+d}}$ $(i = 1, 2)$. Thus the second Lyapunov exponent is negative.

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In 1993, J. C. Lagarias has shown how to use multiplicative ergodic theorems to determine the approximation exponent $1 + d$ for multidimensional continued fractions. Ito, Keane & Ohtsuki 1993 proved that for the two-dimensional modified Jacobi-Perron algorithm $d > 0$ (see also Fujita, Ito, Keane & Ohtsuki 1996). In Schweiger 1996, a classical result of Paley & Ursell 1930 was used to determine the exponent of convergence of the Jacobi-Perron algorithm in two dimensions. Meester 1997 gave another proof for the result on Podsypanin's modification.

In this paper, we will show that a similar method can be applied to Brun's algorithm in two dimensions. Clearly, this is no surprise since Brun's multiplicative algorithm is a factor of the modified algorithm.

We will start with a description of the algorithm; for general references on Brun's algorithm see e.g. Schweiger 1995.

Definition. *Let* $B = \{(x_1, x_2) : 1 \geq x_1 \geq x_2 \geq 0\}$; *Brun's Algorithm is generated by a map* $S : B(j, N) \to B$, *where*

$$S(x_1, x_2) = \begin{cases} \left(\frac{1}{x_1} - N, \frac{x_2}{x_1}\right) & \text{if } \frac{1}{x_1} - N \geq \frac{x_2}{x_1} \quad [j = 1] \\ \left(\frac{x_2}{x_1}, \frac{1}{x_1} - N\right) & \text{if } \frac{1}{x_1} - N < \frac{x_2}{x_1} \quad [j = 2] \end{cases}$$

$$N := \left[\frac{1}{x_1}\right] \geq 1.$$

Let $N^{(t)} := N(S^{t-1}(x_1^{(0)}, x_2^{(0)}))$ and $j(t) := j(S^{t-1}(x_1^{(0)}, x_2^{(0)}))$;

if $j(t+1) = 1$, then $x_1^{(t+1)} = \dfrac{1}{x_1^{(t)}} - N^{(t+1)}, x_2^{(t+1)} = \dfrac{x_2^{(t)}}{x_1^{(t)}}$;

if $j(t+1) = 2$, then $x_1^{(t+1)} = \dfrac{x_2^{(t)}}{x_1^{(t)}}, x_2^{(t+1)} = \dfrac{1}{x_1^{(t)}} - N^{(t+1)}$.

The matrices of Brun's Algorithm are given as follows:

Definition. *Let* $t \geq 1$;

$$\Lambda_B^{(t)} := \begin{pmatrix} N^{(t)} & 2 - j(t) & j(t) - 1 \\ 1 & 0 & 0 \\ 0 & j(t) - 1 & 2 - j(t) \end{pmatrix},$$

then

$$\Omega_B^{(1)} = \begin{pmatrix} q^{(1)} & q^{(0)} & q^{(-1)} \\ p_1^{(1)} & p_1^{(0)} & p_1^{(-1)} \\ p_2^{(1)} & p_2^{(0)} & p_2^{(-1)} \end{pmatrix} := E,$$

and, for $t \geq 2$,

$$\Omega_B^{(t)} = \begin{pmatrix} q^{(t)} & q^{(t')} & q^{(t'')} \\ p_1^{(t)} & p_1^{(t')} & p_1^{(t'')} \\ p_2^{(t)} & p_2^{(t')} & p_2^{(t'')} \end{pmatrix} := \Omega_B^{(t-1)} \Lambda_B^{(t-1)}. \qquad (1)$$

Hence, for $i = 1, 2$, we get

$$x_i^{(0)} = \frac{p_i^{(t)} + x_1^{(t)} p_i^{(t')} + x_2^{(t)} p_i^{(t'')}}{q^{(t)} + x_1^{(t)} q^{(t')} + x_2^{(t)} q^{(t'')}}. \tag{2}$$

Define t^* as the largest integer such that $t^* < t$, and $j(t^*) = 2$ (if there is no such $t^* < t$, then $t^* := -1$); consequently, $(t+1)^*$ is defined as the largest integer such that $(t+1)^* < t+1$, and $j((t+1)^*) = 2$. Then if $j(t) = 1$, $(t+1)'$ in Definition (1) equals t, and $(t+1)'' = t^* = (t+1)^*$; in the other case, we have $(t+1)' = t^*$, and $(t+1)'' = t = (t+1)^*$. Hence

$$\text{if } j(t) = 1\colon q^{(t+2)} = N^{(t+1)} q^{(t+1)} + q^{(t)}, \text{ and} \tag{3}$$

$$\text{if } j(t) = 2\colon q^{(t+2)} = N^{(t+1)} q^{(t+1)} + q^{(t^*)}. \tag{4}$$

Of course, (3) and (4) remain valid if we replace $q^{(\cdot)}$ by $p_i^{(\cdot)}$ for $i = 1, 2$. Since the following results hold for both $p_1^{(\cdot)}$ and $p_2^{(\cdot)}$, from now on we will only write $p^{(\cdot)}$ instead. We continue with a modification of the arguments of Paley & Ursell 1930.

Definition.

$$P_{t+1} := \begin{vmatrix} q^{(t+1)} & q^{(t)} \\ p^{(t+1)} & p^{(t)} \end{vmatrix}, \ P'_{t+1} := \begin{vmatrix} q^{(t+1)} & q^{(t^*)} \\ p^{(t+1)} & p^{(t^*)} \end{vmatrix}, \ P''_{t+1} := \begin{vmatrix} q^{(t)} & q^{(t^*)} \\ p^{(t)} & p^{(t^*)} \end{vmatrix}.$$

By Eqs. (3) and (4) we get the following relations:

$$\text{if } j(t) = 1\colon \ P_{t+2} = -P_{t+1}, \ P'_{t+2} = N^{(t+1)} P'_{t+1} - P''_{t+1}, \ P''_{t+2} = P'_{t+1}; \tag{5}$$

$$\text{if } j(t) = 2\colon \ P_{t+2} = -P'_{t+1}, \ P'_{t+2} = N^{(t+1)} P_{t+1} - P''_{t+1}, \ P''_{t+2} = P_{t+1}. \tag{6}$$

Definition.

$$\rho_t := \max\left\{\frac{|P_t|}{q^{(t)}}, \frac{|P'_t|}{q^{(t)}}\right\} \tag{7}$$

Lemma.

$$|P_{t+1}| \leq q^{(t)} \rho_t \tag{8}$$

Proof: We use (5), (6) and Definition (7): $|P_{t+1}| \leq \max\{|P_t|, |P'_t|\} \leq q^{(t)} \rho_t$.

Lemma.

$$|P''_{t+1}| \leq q^{(t)}\rho_t \tag{9}$$

Proof: Similar to the previous lemma: $|P''_{t+1}| \leq \max\{|P_t|, |P'_t|\} \leq q^{(t)}\rho_t$.

Definition.

$$B_2 := B \cap ((x_1, x_2) : j(x_1, x_2) = 2)$$

$$M := \bigcap_{i=0}^{2} S^{-i} B_2$$

Let $t_0 := \min\{t > 0 : S^{t-1}(x_1, x_2) \in M\}$, $t_{m+1} := \min\{t > t_m : S^{t-1}(x_1, x_2) \in M\}$; we thus have $j(t_m) = 2, j(t_m + 1) = 2$ and $j(t_m + 2) = 2$. Hence, in choosing $(x_1, x_2) \in M$, we avoid t^* being too far away from t, which will simplify the following estimates.

Lemma.

$$\mu(M) > 0$$

Proof: We consider the subset $M^* \subseteq M$, where for $(x_1, x_2) \in M^*$, $N(x_1, x_2) = 1$, $N(S(x_1, x_2)) = 1$ and $N(S^2(x_1, x_2)) = 1$; clearly, $\mu(M) \geq \mu(M^*)$. Since all the cylinders $B(j, N)$ are proper, i.e. $S(B(j, N)) = B$ (see e.g. Schweiger 1998), we can apply the local inverse

$$V_{(j=2,N=1)}(y_1, y_2) = \left(\frac{1}{1+y_2}, \frac{y_1}{1+y_2}\right)$$

to the points (0,0), (1,0) and (1.1). We get a triangle whose vertices are given by the points $V^3(0,0) = \left(\frac{1}{2}, \frac{1}{2}\right)$, $V^3(1,0) = \left(\frac{2}{3}, \frac{1}{3}\right)$ and $V^3(1,1) = \left(\frac{3}{4}, \frac{1}{2}\right)$, which clearly is of positive measure.

Lemma.

$$\rho_{t_m+4} \leq \left(1 - \frac{q^{(t_m)}}{q^{(t_m+4)}}\right) \max\{\rho_{t_m+3}, \rho_{t_m+2}, \rho_{t_m+1}\} \tag{10}$$

Proof: We apply Lemma (8) and Eq. (4) in (1), and similarly relation (6), (8) and (4) in (2)

(1) $|P_{t_m+4}| \leq q^{(t_m+3)}\rho_{t_m+3} \leq (q^{(t_m+4)} - q^{(t_m)})\rho_{t_m+3}$

(2) $$\begin{aligned} |P'_{t_m+4}| &\leq N^{(t_m+3)}|P_{t_m+3}| + |P''_{t_m+3}| \\ &\leq N^{(t_m+3)}|P_{t_m+3}| + |P_{t_m+2}| \\ &\leq (N^{(t_m+3)}q^{(t_m+2)} + q^{(t_m+1)})\max\{\rho_{t_m+2}, \rho_{t_m+1}\} \\ &\leq (N^{(t_m+3)}N^{(t_m+2)}q^{(t_m+2)} + N^{(t_m+3)}q^{(t_m)} + q^{(t_m+1)} - q^{(t_m)}) \\ &\quad \max\{\rho_{t_m+2}, \rho_{t_m+1}\} \\ &\leq (N^{(t_m+3)}q^{(t_m+3)} + q^{(t_m+1)} - q^{(t_m)})\max\{\rho_{t_m+2}, \rho_{t_m+1}\} \\ &\leq (q^{(t_m+4)} - q^{(t_m)})\max\{\rho_{t_m+2}, \rho_{t_m+1}\} \end{aligned}$$

Lemma.

$$\rho_{t_m+5} \leq \left(1 - \frac{q^{(t_m)}}{q^{(t_m+5)}}\right)\max\{\rho_{t_m+3}, \rho_{t_m+2}, \rho_{t_m+1}\}$$

Proof: In (1) we use (5), (6) and the previous lemma; (2.1) follows from (5), the previous lemma, Lemma (9) and Eq. (3)

(1) $$|P_{t_m+5}| \leq \max\{|P_{t_m+4}|, |P'_{t_m+4}|\} \leq (q^{(t_m+4)} - q^{(t_m)}) \max\{\rho_{t_m+3}, \rho_{t_m+2}, \rho_{t_m+1}\}$$

(2.1) $j(t_m + 3) = 1:$

$$\begin{aligned} |P'_{t_m+5}| &\leq N^{(t_m+4)}|P'_{t_m+4}| + |P''_{t_m+4}| \\ &\leq (N^{(t_m+4)}q^{(t_m+4)} - q^{(t_m)})\max\{\rho_{t_m+3}, \rho_{t_m+2}, \rho_{t_m+1}\} \\ &\quad + q^{(t_m+3)}\rho_{t_m+3} \\ &\leq (N^{(t_m+4)}q^{(t_m+4)} + q^{(t_m+3)} - q^{(t_m)}) \\ &\quad \max\{\rho_{t_m+3}, \rho_{t_m+2}, \rho_{t_m+1}\} \\ &\leq (q^{(t_m+5)} - q^{(t_m)})\max\{\rho_{t_m+3}, \rho_{t_m+2}, \rho_{t_m+1}\} \end{aligned}$$

(2.2) $j(t_m + 3) = 2$: Similar to (2) in the previous lemma

Since Lemma [10] guarantees that $\rho_{t_m+4} \leq \max\{\rho_{t_m+3}, \rho_{t_m+2}, \rho_{t_m+1}\}$, for $t_m + 6$ we similarly get the following result:

Lemma.

$$\rho_{t_m+6} \leq \left(1 - \frac{q^{(t_m)}}{q^{(t_m+6)}}\right) \max\{\rho_{t_m+3}, \rho_{t_m+2}, \rho_{t_m+1}\}$$

Now let $t > t_m + 6$; we have

Lemma.

$$\rho_t \leq \left(1 - \frac{q^{(t_m)}}{q^{(t_m+6)}}\right) \max\{\rho_{t_m+3}, \rho_{t_m+2}, \rho_{t_m+1}\}$$

Proof: We proceed inductively, where the assumptions are given by the previous lemmas; (1) follows from (5), (6) and Definition (7); in (2.1) we apply (5), (9) and (3), in (2.2.1) we use (6), (5), Lemma (9), (3) and (4), in (2.2.2) (6), Definition (7), Lemma (9) and (4)

(1)
$$\frac{|P_t|}{q^{(t)}} \leq \frac{\max\{|P_{t-1}|, |P'_{t-1}|\}}{q^{(t-1)}} \leq \rho_{t-1}$$

$$\leq \left(1 - \frac{q^{(t_m)}}{q^{(t_m+6)}}\right) \max\{\rho_{t_m+3}, \rho_{t_m+2}, \rho_{t_m+1}\}$$

(2.1) $j(t-2) = 1$:

$$\frac{|P'_t|}{q^{(t)}} \leq \frac{N^{(t-1)}|P'_{t-1}| + |P''_{t-1}|}{q^{(t)}}$$

$$\leq \frac{(N^{(t-1)}q^{(t-1)} + q^{(t-2)})\max\{\rho_{t-1}, \rho_{t-2}\}}{q^{(t)}}$$

$$\leq \max\{\rho_{t-1}, \rho_{t-2}\}$$

$$\leq \left(1 - \frac{q^{(t_m)}}{q^{(t_m+6)}}\right) \max\{\rho_{t_m+3}, \rho_{t_m+2}, \rho_{t_m+1}\}$$

(2.2) $j(t-2) = 2$:

(2.2.1) $j(t-3) = 1$:

$$\begin{aligned}
\frac{|P'_t|}{q^{(t)}} &\leq \frac{N^{(t-1)}|P_{t-1}| + |P''_{t-1}|}{q^{(t)}} \\
&\leq \frac{N^{(t-1)}|P_{t-2}| + |P''_{t-1}|}{q^{(t)}} \\
&\leq \frac{(N^{(t-1)}q^{(t-3)} + q^{(t-2)})\max\{\rho_{t-2}, \rho_{t-3}\}}{q^{(t)}} \\
&\leq \frac{(N^{(t-1)}N^{(t-2)}q^{(t-2)} + N^{(t-1)}q^{(t-3)})\max\{\rho_{t-2}, \rho_{t-3}\}}{q^{(t)}} \\
&\leq \frac{N^{(t-1)}q^{(t-1)}\max\{\rho_{t-2}, \rho_{t-3}\}}{q^{(t)}} \\
&\leq \max\{\rho_{t-2}, \rho_{t-3}\} \\
&\leq \left(1 - \frac{q^{(t_m)}}{q^{(t_m+6)}}\right)\max\{\rho_{t_m+3}, \rho_{t_m+2}, \rho_{t_m+1}\}
\end{aligned}$$

(2.2.2) $j(t-3) = 2$:

$$\begin{aligned}
\frac{|P'_t|}{q^{(t)}} &\leq \frac{N^{(t-1)}|P_{t-1}| + |P''_{t-1}|}{q^{(t)}} \\
&\leq \frac{N^{(t-1)}|P_{t-1}| + |P_{t-2}|}{q^{(t)}} \\
&\leq \frac{(N^{(t-1)}q^{(t-1)} + q^{(t-3)})\max\{\rho_{t-1}, \rho_{t-3}\}}{q^{(t)}} \\
&\leq \max\{\rho_{t-1}, \rho_{t-3}\} \\
&\leq \left(1 - \frac{q^{(t_m)}}{q^{(t_m+6)}}\right)\max\{\rho_{t_m+3}, \rho_{t_m+2}, \rho_{t_m+1}\}
\end{aligned}$$

We get the following

Lemma. *Let* $\tau_m := \max\{\rho_{t_m+3}, \rho_{t_m+2}, \rho_{t_m+1}\}$; *then*

$$\tau_{m+3} \leq \left(1 - \frac{q^{(t_m)}}{q^{(t_m+6)}}\right)\tau_m. \tag{11}$$

We can now use the quantities ρ_t and τ_m to estimate the approximation quality:

Lemma.

$$\left|x_i - \frac{p_i^{(t)}}{q^{(t)}}\right| \le \frac{2\rho_t}{q^{(t)}} \tag{12}$$

Proof: Recall (2):

$$x_i^{(0)} = \frac{p_i^{(t)} + x_1^{(t)} p_i^{(t')} + x_2^{(t)} p_i^{(t'')}}{q^{(t)} + x_1^{(t)} q^{(t')} + x_2^{(t)} q^{(t'')}};$$

Hence

$$\begin{aligned}
\left|x_i - \frac{p_i^{(t)}}{q^{(t)}}\right| &\le \left|\frac{p_i^{(t)} + x_1^{(t)} p_i^{(t')} + x_2^{(t)} p_i^{(t'')}}{q^{(t)} + x_1^{(t)} q^{(t')} + x_2^{(t)} q^{(t'')}} - \frac{p_i^{(t)}}{q^{(t)}}\right| \\
&\le \left|\frac{q^{(t)} p_i^{(t)} + x_1^{(t)} q^{(t)} p_i^{(t')} + x_2^{(t)} q^{(t)} p_i^{(t'')}}{\left(q^{(t)}\right)^2}\right. \\
&\qquad \left. - \frac{q^{(t)} p_i^{(t)} + x_1^{(t)} q^{(t')} p_i^{(t)} + x_2^{(t)} q^{(t'')} p_i^{(t)}}{\left(q^{(t)}\right)^2}\right| \\
&\le \frac{x_1^{(t)} |q^{(t)} p_i^{(t')} - q^{(t')} p_i^{(t)}| + x_2^{(t)} |q^{(t)} p_i^{(t'')} - q^{(t'')} p_i^{(t)}|}{\left(q^{(t)}\right)^2} \\
&\le \frac{2\rho_t}{q^{(t)}}
\end{aligned}$$

Theorem. For almost all $(x_1, x_2) \in B$ there is a constant $d > 0$ such that

$$\left|x_i - \frac{p_i^{(t)}}{q^{(t)}}\right| \le \frac{1}{\left(q^{(t)}\right)^{1+d}}.$$

Proof: We will first consider the case that $(x_1, x_2) \in M$. By (3), (4) we know that

$$q^{(t_m+6)} \le 16 N^{(t_m+5)} N^{(t_m+4)} N^{(t_m+3)} N^{(t_m+2)} N^{(t_m+1)} N^{(t_m)} q^{(t_m)}.$$

Hence

$$1 - \frac{q^{(t_m)}}{q^{(t_m+6)}} \le 1 - \frac{1}{16 N^{(t_m+5)} N^{(t_m+4)} N^{(t_m+3)} N^{(t_m+2)} N^{(t_m+1)} N^{(t_m)}}.$$

Define

$$f(x_1, x_2) := \log\left(1 - \frac{1}{16N^{(6)}N^{(5)}N^{(4)}N^{(3)}N^{(2)}N^{(1)}}\right).$$

Let $(x_1, x_2) \in M$; define the return time $r_M(x_1, x_2) := \min\{k > 0 : S^k(x_1, x_2) \in M\}$, and the induced transformation

$$S_M : M \to M, S_M := S^{r_M(x_1, x_2)}.$$

We then have

$$S_M^m(x_1, x_2) = S^{t_m - 1}(x_1, x_2).$$

Since Brun's Algorithm is ergodic and conservative (for a proof see e.g. Schweiger 1998), so is the system (M, S_M, μ); we can apply the ergodic theorem and get

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_{i=0}^{m} f(S_M^i(x_1, x_2)) = \frac{\int_M f(x_1, x_2) d\mu}{\mu(M)} =: \frac{\log K_1}{\mu(M)} < 0.$$

Thus by (11), for m large enough,

$$\tau_m \leq c K_1^{\frac{m}{3\mu(M)}}.$$

Since

$$\lim_{m\to\infty} \frac{t_m}{m} \to \frac{1}{\mu(M)},$$

$$\tau_m \leq c K_1^{\frac{t_m}{3}},$$

and, for t large enough,

$$\rho_t \leq c K_1^{\frac{t}{3}}.$$

We have $\mu(M) > 0$, hence (since S is conservative) t_0 is finite a.e., and we can generalize the result to $(x_1, x_2) \in B$.

On the other hand, by (3) and (4) we estimate

$$q^{(t+1)} \leq 2N^{(t)} q^{(t)} \leq \frac{2}{x_1^{(t)}} q^{(t)},$$

and define

$$g(x_1, x_2) := \log \frac{x_1}{2}.$$

We have

$$\lim_{t\to\infty} \frac{1}{t}\sum_{i=1}^{t} g(S^i(x_1, x_2)) =: -\log K_2 < 0,$$

and

$$q^{(t)} \leq K_2^t.$$

Therefore

$$\rho_t \leq \frac{1}{\left(q^{(t)}\right)^{d'}},$$

and by Lemma (12)

$$\left| x_i - \frac{p_i^{(t)}}{q^{(t)}} \right| \leq \frac{1}{\left(q^{(t)}\right)^{1+d}} \quad \text{a.e.}$$

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Author's address: Mag. B. R. Schratzberger, Institut für Mathematik, Universität Salzburg, Hellbrunnerstr. 34, A-5020 Salzburg.