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Invariant Measure and Exponent of Convergence for Baldwin's Algorithm GCFP

By

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Abstract

During the last years ergodic properties of several multidimensional continued fractions have been investigated (Schweiger 2000). In Baldwin 1992a and 1992b the generalized mediant algorithm (which is equivalent to Selmer's algorithm by a shift of coordinates) is studied in greater detail. Furthermore an algorithm, called the GCFP algorithm, is mentioned (Baldwin 1992b p. 1517). Baldwin states that "An analytical form for the invariant measure and entropy of GCFP are unknown". The purpose of this note is to calculate the invariant measure for this algorithm and to show that the convergence exponent is positive almost everywhere.

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1.

Selmer's algorithm in any dimension can be defined as follows: Let $\Delta^{n+1} := \{b = (b_0, b_1, \dots, b_n) : b_0 \geq b_1 \geq \cdots \geq b_n \geq 0\}$. Then we define

$$\sigma b := (b_0 - b_n, b_1, \dots, b_n).$$

There is an index $i = i(b)$, $0 \leq i \leq n$, such that

$$\pi\sigma b := (b_1, b_2, \dots, b_i, b_0 - b_n, \dots, b_n) \in \Delta^{n+1}.$$

With the help of the projection $p : \Delta^{n+1} \to \Delta^n$ defined by $p(b_0, b_1, \ldots, b_n) = \left(\frac{b_1}{b_0}, \ldots, \frac{b_n}{b_0}\right)$ we get a *bottom map* T with the property

$$p \circ \pi \circ \sigma = T \circ p.$$

In this paper we consider only the case $n = 2$. It is well known that the set $\Delta := \{(x, y) : 0 \leq 1 - x \leq y \leq x \leq 1\}$ is *absorbing* (i.e. almost every $(x,\ y)$ eventually enters Δ and stays there forever).

Therefore the ergodic behaviour of *Selmer's algorithm* can be obtained by the study of the following map:

$$T(x, y) = \left(\frac{1 - y}{x}, \frac{y}{x}\right) \text{ if } (x, y) \in \Delta(1) := \{(x, y) \in \Delta : 2y \leq 1\}$$
$$T(x, y) = \left(\frac{y}{x}, \frac{1 - y}{x}\right) \text{ if } (x, y) \in \Delta(2) := \{(x, y) \in \Delta : 1 \leq 2y\}.$$

We define the *first entry time* of the set $\Delta(2)$ almost everywhere on Δ as

$$e(x, y) := \min\{k \geq 0 : T^k(x, y) \in \Delta(2)\}.$$

The map $S : \Delta \to \Delta,\ S(x, y) := T^{e(x,y)+1}(x, y)$ is the *jump transformation* of T with respect to the set $\Delta(2)$ (see Schweiger 1995).

The map S is equivalent to Baldwin's GCFP algorithm. The time-1-partition is defined by the following triangles:

$$\Delta(t\alpha) \text{ has the vertices } \left(1, \frac{1}{1 + t}\right), \left(\frac{1 + t}{2 + t}, \frac{1}{2 + t}\right), \left(1, \frac{1}{2 + t}\right)$$
$$\Delta(t\beta) \text{ has the vertices } \left(\frac{1 + t}{2 + t}, \frac{1}{2 + t}\right), \left(1, \frac{1}{2 + t}\right), \left(\frac{2 + t}{3 + t}, \frac{1}{3 + t}\right).$$

Here $t = \left[\frac{1}{y}\right] - 1$ on $\Delta(t\alpha)$ and $t = \left[\frac{1}{y}\right] - 2$ on $\Delta(t\beta)$. Then one verifies the explicit version of S as follows:

Theorem 1

$$S(x, y) = \left(\frac{y}{x - ty}, \frac{1 - (t + 1)y}{x - ty}\right), (x, y) \in \Delta(t\alpha)$$
$$S(x, y) = \left(\frac{y}{1 - (t + 1)y}, \frac{x - (t + 1)y}{1 - (t + 1)y}\right), (x, y) \in \Delta(t\beta).$$

Remark. The local inverse branches are given as

$$V(t\alpha)(x,y) = \left(\frac{1+tx}{(1+t)x+y}, \frac{x}{(1+t)x+y}\right)$$
$$V(t\beta)(x,y) = \left(\frac{(1+t)x+y}{1+(1+t)x}, \frac{x}{1+(1+t)x}\right).$$

Theorem 2. *The invariant density is given by*

$$\sigma(x,y) = \frac{1}{x(x+y)}.$$

Proof: If one has already found the shape of $\sigma(x,y)$ then Kuzmin's equation can be verified quite easily.

$$\sum_{t=0}^{\infty}\left(\sigma\left(\frac{1+tx}{(1+t)x+y}, \frac{x}{(1+t)x+y}\right)\frac{1}{((1+t)x+y)^3}+\right.$$
$$\left. +\,\sigma\left(\frac{(1+t)x+y}{1+(1+t)x}, \frac{x}{1+(1+t)x}\right)\frac{1}{(1+(1+t)x)^3}\right)$$
$$=\sum_{t=0}^{\infty}\left(\frac{1}{(1+tx)(1+(t+1)x)((1+t)x+y)}+\right.$$
$$\left. +\,\frac{1}{((1+t)x+y))((2+t)x+y)(1+(1+t)x)}\right)$$
$$=\sum_{t=0}^{\infty}\frac{1}{x}\left(\frac{1}{(1+tx)((1+t)x+y)} - \frac{1}{(1+(1+t)x)((2+t)x+y)}\right)$$
$$=\frac{1}{x(x+y)}.$$

The more interesting question is how to find this density! Here the connection between jump transformations and first return maps is helpful.

The *first return time* of the set $\Delta(2)$ is defined almost everywhere on $\Delta(2)$ as

$$n(x,y) := \min\{n \geq 1 : T^n(x,y) \in \Delta(2)\}.$$

The map $R : \Delta(2) \to \Delta(2)$, $R(x,y) := T^{n(x,y)}(x,y)$ is known as the *first return map*.

Let μ denote the invariant measure of T with density τ and ν denote the invariant measure for S with density σ. It is well known that the restriction of μ on measurable subsets of $\Delta(2)$ is an invariant measure for R.

Lemma 1. *The dynamical systems* $(\Delta(2), R, \mu)$ *and* (Δ, S, ν) *are isomorphic. In fact the diagram*

$$\begin{array}{ccc} \Delta(2) & \xrightarrow{R} & \Delta(2) \\ T\downarrow & & \downarrow T \\ \Delta & \underset{S}{\longrightarrow} & \Delta \end{array}$$

is commutative.

Proof: Observe that $e(T(x, y)) + 1 = n(x, y)$. Furthermore $T\colon \Delta(2) \to \Delta$ is bijective.

The relation $\mu(T^{-1}H \cap \Delta(2)) = \nu(H)$ shows that

$$\sigma(x, y) = \tau\left(\frac{1}{x+y}, \frac{x}{x+y}\right)\frac{1}{(x+y)^3}.$$

Since $\tau(x, y) = \frac{1}{xy}$ we find $\sigma(x, y)$ as given in theorem 2.

Remarks:

(1) The normalization constant is given as

$$N(S) = \iint_{\Delta} \sigma(x, y)\,dx\,dy = \int_{\frac{1}{2}}^{1} dx \int_{1-x}^{x} \frac{dy}{x(x+y)}.$$

(2) The entropy can be calculated as

$$h(S) = -\frac{3}{N(S)}\sum_{t=0}^{\infty}\left(\iint_{\Delta(t\alpha)} \log(x - ty)\sigma(x, y)\,dx\,dy + \right.$$
$$\left. + \iint_{\Delta(t\beta)} \log(1 - (t+1)y)\sigma(x, y)\,dx\,dy\right).$$

2.

Define

$$B(t\alpha) := \begin{pmatrix} 0 & t+1 & 1 \\ 1 & t & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B(t\beta) := \begin{pmatrix} 1 & t+1 & 0 \\ 0 & t+1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$M(t_1\varepsilon_1, \ldots, t_s\varepsilon_s, t_{s+1}\varepsilon_{s+1}) := M(t_1\varepsilon_1, \ldots, t_s\varepsilon_s)B(t_{s+1}\varepsilon_{s+1})$$
$$0 \leq t_i, \quad \varepsilon_i \in \{\alpha, \beta\}, \quad i = 1, \ldots, s+1$$

$$M(t_1\varepsilon_1,\ldots,t_s\varepsilon_s) =: \begin{pmatrix} B_{00}^{(s)} & B_{01}^{(s)} & B_{02}^{(s)} \\ B_{10}^{(s)} & B_{11}^{(s)} & B_{12}^{(s)} \\ B_{20}^{(s)} & B_{21}^{(s)} & B_{22}^{(s)} \end{pmatrix}$$

$$[i,j]_1^{(s)} := \begin{vmatrix} B_{0i}^{(s)} & B_{0j}^{(s)} \\ B_{1i}^{(s)} & B_{1j}^{(s)} \end{vmatrix}$$

$$[i,j]_2^{(s)} := \begin{vmatrix} B_{0i}^{(s)} & B_{0j}^{(s)} \\ B_{2i}^{(s)} & B_{2j}^{(s)} \end{vmatrix}$$

In the sequel we write $[i,j]^{(s)}$ to mean $[i,j]_1^{(s)}$ or $[i,j]_2^{(s)}$. We also define $B_{ij}^{(0)} := \delta_{ij}$.

Lemma 2. *The following recursion relations are valid.*

For brevity we write $t = t_{s+1}$

$\boxed{\varepsilon_{s+1} = \alpha}$ $\quad i = 1,2,3$

$$\begin{aligned} B_{i0}^{(s+1)} &= B_{i1}^{(s)} \\ B_{i1}^{(s+1)} &= (t+1)B_{i0}^{(s)} + tB_{i1}^{(s)} + B_{i2}^{(s)} \\ B_{i2}^{(s+1)} &= B_{i0}^{(s)} \end{aligned}$$

$$\begin{aligned} [0,1]^{(s+1)} &= -(t+1)[0,1]^{(s)} + [1,2]^{(s)} \\ [0,2]^{(s+1)} &= -[0,1]^{(s)} \\ [1,2]^{(s+1)} &= -t[0,1]^{(s)} - [0,2]^{(s)} \end{aligned}$$

$\boxed{\varepsilon_{s+1} = \beta}$ $\quad i = 1,2,3$

$$\begin{aligned} B_{i0}^{(s+1)} &= B_{i0}^{(s)} \\ B_{i1}^{(s+1)} &= (t+1)B_{i0}^{(s)} + (t+1)B_{i1}^{(s)} + B_{i2}^{(s)} \\ B_{i2}^{(s+1)} &= B_{i1}^{(s)} \end{aligned}$$

$$\begin{aligned} [0,1]^{(s+1)} &= (t+1)[0,1]^{(s)} + [0,2]^{(s)} \\ [0,2]^{(s+1)} &= [0,1]^{(s)} \\ [1,2]^{(s+1)} &= (t+1)[0,1]^{(s)} - [1,2]^{(s)} \end{aligned}$$

Lemma 3. $\boxed{\varepsilon_{s+3} = \varepsilon_{s+2} = \varepsilon_{s+1} = \alpha}$
At least one of the three consecutive products $[0,1]^{(s+j)}[1,2]^{(s+j)}$, $j = 0,1,2$ *is nonnegative.*

Proof: We may assume $[0,1]^{(s)} \geq 0$.

(1) If $[1,2]^{(s)} \geq 0$ we are done.
(2) Suppose $[1,2]^{(s)} \leq 0$ and $[0,2]^{(s)} \geq 0$. Then

$$[0,1]^{(s+1)} = -(t+1)[0,1]^{(s)} + [1,2]^{(s)} \leq 0$$

$$[0,2]^{(s+1)} = -[0,1]^{(s)} \leq 0$$

$$[1,2]^{(s+1)} = -t[0,1]^{(s)} - [0,2]^{(s)} \leq 0.$$

Then clearly $[0,1]^{(s+1)}\ [1,2]^{(s+1)} \geq 0$.
(3) Suppose $[1,2]^{(s)} \leq 0$ and $[0,2]^{(s)} \leq 0$. Then $[0,1]^{(s+1)} = -(t+1)[0,1]^{(s)} + [1,2]^{(s)} \leq 0$
and $[0,2]^{(s+1)} = -[0,1]^{(s)} \leq 0$.
If $[1,2]^{(s+1)} = -t[0,1]^{(s)} - [0,2]^{(s)} \leq 0$ we obtain $[0,1]^{(s+1)}$ $[1,2]^{(s+1)} \geq 0$.
If $[1,2]^{(s+1)} = -t[0,1]^{(s)} - [0,2]^{(s)} \geq 0$ then this is case (2) with signs reversed. Therefore $[0,1]^{(s+2)}\ [1,2]^{(s+2)} \geq 0$.

Lemma 4

$$|[0,1]^{(s)}| \leq B_{00}^{(s)} + B_{01}^{(s)}$$

$$|[0,2]^{(s)}| \leq B_{00}^{(s)} + B_{02}^{(s)}$$

$$|[1,2]^{(s)}| \leq B_{01}^{(s)} + B_{02}^{(s)}$$

Proof: The cases $s = 0,1,2$ are verified by inspection. We proceed by induction.

$\boxed{\varepsilon_{s+1} = \beta}$

$$|[0,2]^{(s+1)}| = |[0,1]^{(s)}| \leq B_{00}^{(s)} + B_{01}^{(s)} \leq B_{00}^{(s+1)} + B_{02}^{(s+1)}$$

$$\begin{aligned} |[1,2]^{(s+1)}| &\leq (t+1)|[0,1]^{(s)}| + |[1,2]^{(s)}| \\ &\leq (t+1)(B_{00}^{(s)} + B_{01}^{(s)}) + B_{01}^{(s)} + B_{02}^{(s)} \\ &= B_{01}^{(s+1)} + B_{02}^{(s+1)} \end{aligned}$$

$$\begin{aligned}|[0,1]^{(s+1)}| &\le (t+1)|[0,1]^{(s)}| + |[0,2]^{(s)}|\\ &\le (t+1)(B_{00}^{(s)} + B_{01}^{(s)}) + B_{00}^{(s)} + B_{02}^{(s)}\\ &= B_{00}^{(s+1)} + B_{01}^{(s+1)}\end{aligned}$$

$\boxed{\varepsilon_{s+1} = \alpha}$

$$|[0,2]^{(s+1)}| = |[0,1]^{(s)}| \le B_{00}^{(s)} + B_{01}^{(s)} = B_{00}^{(s+1)} + B_{02}^{(s+1)}$$

$$\begin{aligned}|[1,2]^{(s+1)}| &\le t|[0,1]^{(s)}| + |[0,2]^{(s)}|\\ &\le t(B_{00}^{(s)} + B_{01}^{(s)}) + B_{00}^{(s)} + B_{02}^{(s)}\\ &= B_{01}^{(s+1)} = (B_{01}^{(s+1)} + B_{02}^{(s+1)})\left(1 - \frac{B_{00}^{(s)}}{B_{01}^{(s+1)} + B_{02}^{(s+1)}}\right)\end{aligned}$$

(1) Let $[0,1]^{(s)}[1,2]^{(s)} \ge 0$. Then

$$\begin{aligned}|[0,1]^{(s+1)}| &\le \max((t+1)|[0,1]^{(s)}|, |[1,2]^{(s)}|)\\ &\le \max((t+1)(B_{00}^{(s)} + B_{01}^{(s)}), B_{01}^{(s)} + B_{02}^{(s)})\\ &\le B_{00}^{(s+1)} + B_{01}^{(s+1)}\end{aligned}$$

If additionally $t \ge 1$ then clearly

$$\max((t+1)(B_{00}^{(s)} + B_{01}^{(s)}), B_{01}^{(s)} + B_{02}^{(s)})) = (t+1)(B_{00}^{(s)} + B_{01}^{(s)})$$

and we obtain the better estimate

$$|[0,1]^{(s+1)}| \le B_{00}^{(s+1)} + B_{01}^{(s+1)}\left(1 - \frac{B_{02}^{(s)}}{B_{00}^{(s+1)} + B_{01}^{(s+1)}}\right)$$

(2) Let $[0,1]^{(s)}[1,2]^{(s)} \le 0$.

$\boxed{\varepsilon_{s+1} = \alpha,\ \varepsilon_s = \beta}$

$$\begin{aligned}[0,1]^{(s+1)} &= -(t_{s+1}t_s + t_{s+1})[0,1]^{(s-1)}\\ &\quad - (t_{s+1}+1)[0,2]^{(s-1)} - [1,2]^{(s-1)}.\end{aligned}$$

Therefore

$$\begin{aligned}|[0,1]^{(s+1)}| &\le (t_{s+1}t_s + t_{s+1})(B_{00}^{(s-1)} + B_{01}^{(s-1)}) +\\ &\quad + (t_{s+1}+1)(B_{00}^{(s-1)} + B_{02}^{(s-1)}) + (B_{01}^{(s-1)} + B_{02}^{(s-1)}).\end{aligned}$$

On the other hand

$$B_{00}^{(s+1)}+B_{01}^{(s+1)}=(t_{s+1}t_s+t_{s+1})(B_{00}^{(s-1)}+B_{01}^{(s-1)})+ \\ +(t_s+1)(B_{00}^{(s-1)}+B_{01}^{(s-1)})+ \\ +(t_{s+1}+1)(B_{00}^{(s-1)}+B_{02}^{(s-1)})+B_{01}^{(s-1)}.$$

Since $B_{02}^{(s-1)} \leq B_{01}^{(s-1)}$ we obtain

$$|[0,1]^{(s+1)}| \leq (B_{00}^{(s+1)}+B_{01}^{(s+1)})\left(1-\frac{B_{00}^{(s-1)}}{B_{00}^{(s+1)}+B_{01}^{(s+1)}}\right).$$

$\boxed{\varepsilon_{s+1}=\varepsilon_s=\alpha}$

$$[0,1]^{(s+1)}=(t_{s+1}t_s+t_{s+1}+1)[0,1]^{(s-1)} \\ -(t_{s+1}+1)[1,2]^{(s-1)}-[0,2]^{(s-1)}.$$

$\boxed{\varepsilon_{s+1}=\varepsilon_s=\alpha,\ \varepsilon_{s-1}=\beta}$

$$[0,1]^{(s+1)}=(t_{s+1}t_st_{s-1}+t_{s+1}t_s-1)[0,1]^{(s-2)}+ \\ +(t_{s+1}t_s+t_{s+1}+1)[0,2]^{(s-2)}+ \\ +(t_{s+1}+1)[1,2]^{(s-2)}$$

$$|[0,1]^{(s+1)}| \leq (t_{s+1}t_st_{s-1}+t_{s+1}t_s)(B_{00}^{(s-2)}+B_{01}^{(s-2)})+ \\ +(t_{s+1}t_s+t_{s+1}+1)(B_{00}^{(s-2)}+B_{02}^{(s-2)})+ \\ +(t_{s+1}+1)(B_{01}^{(s-2)}+B_{02}^{(s-2)})-B_{01}^{(s-2)}$$

$$B_{00}^{(s+1)}+B_{01}^{(s+1)}=(t_{s+1}t_st_{s-1}+t_{s+1}t_s)(B_{00}^{(s-2)}+B_{01}^{(s-2)})+ \\ +(t_{s+1}t_s+t_{s+1}+t_s+1)(B_{00}^{(s-2)}+B_{02}^{(s-2)})+ \\ +(t_{s+1}t_{s-1}+t_st_{s-1}+t_{s+1}+t_s+ \\ +t_{s-1}+1)(B_{00}^{(s-2)}+B_{01}^{(s-2)})+ \\ +(t_{s+1}+1)B_{01}^{(s-2)}+B_{00}^{(s-2)}.$$

Since $B_{02}^{(s-2)} \leq B_{01}^{(s-2)}$ we obtain

$$|[0,1]^{(s+1)}| \leq (B_{00}^{(s+1)}+B_{01}^{(s+1)})\left(1-\frac{B_{01}^{(s-2)}}{B_{00}^{(s+1)}+B_{01}^{(s+1)}}\right).$$

$\boxed{\varepsilon_{s+1}=\varepsilon_s=\varepsilon_{s-1}=\alpha}$

(2.1) Let $[0,1]^{(s-1)}[1,2]^{(s-1)} \geq 0$.

Since $\varepsilon_{s-1} = \alpha$ we can use the better estimate $|[1,2]^{(s-1)}| \leq B_{01}^{(s-1)}$.

Then

$$\begin{aligned}|[0,1]^{(s+1)}| &\leq \max((t_{s+1}t_s + t_{s+1} + 1)|[0,1]^{(s-1)}|,\\ &\qquad (t_{s+1}+1)|[1,2]^{(s-1)}|) + |[0,2]^{(s-1)}|\\ &\leq \max((t_{s+1}t_s + t_{s+1} + 1)(B_{00}^{(s-1)} + B_{01}^{(s-1)}),\\ &\qquad (t_{s+1}+1)B_{01}^{(s-1)}) + B_{00}^{(s-1)} + B_{02}^{(s-1)}\\ &= (t_{s+1}t_s + t_{s+1} + 1)(B_{00}^{(s-1)} + B_{01}^{(s-1)}) +\\ &\quad + B_{00}^{(s-1)} + B_{02}^{(s-1)}.\end{aligned}$$

Since

$$\begin{aligned}B_{00}^{(s+1)} + B_{01}^{(s+1)} &= (t_s t_{s+1} + t_{s+1} + 1)(B_{00}^{(s-1)} + B_{01}^{(s-1)}) +\\ &\quad + t_s(B_{00}^{(s-1)} + B_{01}^{(s-1)}) +\\ &\quad + (t_{s+1}+1)B_{02}^{(s-1)} + B_{00}^{(s-1)}\end{aligned}$$

we obtain

$$|[0,1]^{(s+1)}| \leq (B_{00}^{(s+1)} + B_{01}^{(s+1)})\left(1 - \frac{B_{01}^{(s-1)}}{B_{00}^{(s+1)} + B_{01}^{(s+1)}}\right).$$

(2.2) Let $[0,1]^{(s-1)}[1,2]^{(s-1)} \leq 0$.

If $[0,1]^{(s)}[1,2]^{(s)} \leq 0$ and $[0,1]^{(s-1)}[1,2]^{(s-1)} \leq 0$ then by lemma 3 we get $[0,1]^{(s-2)}[1,2]^{(s-2)} \geq 0$.

$$\begin{aligned}[0,1]^{(s+1)} &= -(t_{s+1}t_s t_{s-1} + t_{s+1}t_s + t_{s+1})[0,1]^{(s-2)} +\\ &\quad + (t_{s+1}t_s + t_{s+1} + 1)[1,2]^{(s-2)} +\\ &\quad + (t_{s+1}+1)[0,2]^{(s-2)}.\end{aligned}$$

Therefore

$$\begin{aligned}|[0,1]^{(s+1)}| &\leq \max((t_{s+1}t_s t_{s-1} + t_{s+1}t_s + t_{s+1})(B_{00}^{(s-2)} + B_{01}^{(s-2)}),\\ &\qquad (t_{s+1}t_s + t_{s+1} + 1)(B_{01}^{(s-2)} + B_{02}^{(s-2)})) +\\ &\quad + (t_{s+1}+1)(B_{00}^{(s-2)} + B_{02}^{(s-2)}).\end{aligned}$$

We calculate

$$\begin{aligned}B_{00}^{(s+1)}+B_{01}^{(s+1)}=(&t_{s+1}t_st_{s-1}+t_{s+1}t_{s-1}+t_st_{s-1}+\\&+t_{s+1}t_s++t_{s+1}+t_s+t_{s-1}+1)+\\&+(B_{00}^{(s-2)}+B_{01}^{(s-2)})+\\&+(t_{s+1}+1)(B_{02}^{(s-2)}+B_{00}^{(s-2)})+\\&+(t_{s+1}t_s+t_s)B_{02}^{(s-2)}+B_{01}^{(s-2)}.\end{aligned}$$

Therefore

$$|[0,1]^{(s+1)}|\le(B_{00}^{(s+1)}+B_{01}^{(s+1)}).$$

However, if $t_s\ge1$ or $t_{s-1}\ge1$, we get the improved estimate

$$|[0,1]^{(s+1)}|\le(B_{00}^{(s+1)}+B_{01}^{(s+1)})\left(1-\frac{B_{01}^{(s-2)}}{B_{00}^{(s+1)}+B_{01}^{(s+1)}}\right).$$

Theorem 3. *For all pairs $(x,y)\in\Delta$ with a nonterminating algorithm the inequalities*

$$\left|x-\frac{B_{11}^{(s)}}{B_{01}^{(s)}}\right|\le\frac{4}{B_{01}^{(s)}}$$

$$\left|y-\frac{B_{21}^{(s)}}{B_{01}^{(s)}}\right|\le\frac{4}{B_{01}^{(s)}}$$

are valid for $s\ge1$.

Proof: Put $(\xi,\eta):=T^s(x,y)$. Then

$$\begin{aligned}\left|x-\frac{B_{11}^{(s)}}{B_{01}^{(s)}}\right|&=\left|\frac{B_{10}^{(s)}+B_{11}^{(s)}\xi+B_{12}^{(s)}\eta}{B_{00}^{(s)}+B_{01}^{(s)}\xi+B_{02}^{(s)}\eta}-\frac{B_{11}^{(s)}}{B_{01}^{(s)}}\right|\\&\le\frac{|[0,1]_1^{(s)}|+|[1,2]_1^{(s)}|\eta}{(B_{00}^{(s)}+B_{01}^{(s)}\xi+B_{02}^{(s)}\eta)B_{01}^{(s)}}.\end{aligned}$$

Since $\frac{1}{2}\le\xi\le1$ and $0\le\eta\le1$ we get

$$|xB_{01}^{(s)}-B_{11}^{(s)}|\le4\ \max\left(\frac{|[0,1]_1^{(s)}|}{B_{00}^{(s)}+B_{01}^{(s)}},\frac{|[1,2]_1^{(s)}|}{B_{01}^{(s)}+B_{02}^{(s)}}\right)\le4.$$

A similar estimate can be given for the second coordinate.

Theorem 4. *There exists a constant $d > 0$ such that for almost all $(x, y) \in \Delta$ the inequalities*

$$\left| x - \frac{B_{11}^{(s)}}{B_{01}^{(s)}} \right| \ll \frac{1}{(B_{01}^{(s)})^{1+d}}$$

$$\left| y - \frac{B_{21}^{(s)}}{B_{01}^{(s)}} \right| \ll \frac{1}{(B_{01}^{(s)})^{1+d}}$$

are valid for $s \geq s_0(x, y)$.

Proof: Let

$$\rho_s(x,y) = \max\left(\frac{|[0,1]_1^{(s)}|}{B_{00}^{(s)} + B_{01}^{(s)}}, \frac{|[1,2]_1^{(s)}|}{B_{01}^{(s)} + B_{02}^{(s)}}, \frac{|[0,2]_1^{(s)}|}{B_{00}^{(s)} + B_{02}^{(s)}} \right).$$

Then the proof of Lemma 4 also shows that $\rho_{s+1} \leq \rho_s$.

Lemma 5. *If* $\boxed{\varepsilon_i = \alpha \text{ and } t_i \geq 1 \text{ for } s-2 \leq i \leq s+3}$, *then*

$$\rho_{s+3}(x,y) \leq \kappa_s(x,y)\rho_s(x,y)$$

where

$$\kappa_s(x,y) = \max\left(1 - \frac{B_{01}^{(s-2)}}{B_{00}^{(s+1)} + B_{01}^{(s+1)}}, 1 - \frac{B_{01}^{(s-1)}}{B_{00}^{(s+2)} + B_{01}^{(s+2)}}, \right.$$
$$\left. 1 - \frac{B_{01}^{(s)}}{B_{00}^{(s+3)} + B_{01}^{(s+3)}}, 1 - \frac{B_{01}^{(s+1)}}{B_{00}^{(s+4)} + B_{01}^{(s+4)}} \right)$$

Proof: We note first that $\varepsilon_{r-1} = \varepsilon_{r-2} = \alpha$ implies $B_{02}^{(r)} = B_{00}^{(r-1)} = B_{01}^{(r-2)}$. Since $\varepsilon_{s+1} = \varepsilon_{s+2} = \varepsilon_{s+3} = \alpha$ then at least once we find for some $j = 0, 1, 2$

$$|[0,1]^{(s+j+1)}| \leq \rho_{s+j}(B_{00}^{(s+j+1)} + B_{01}^{(s+j+1)}) \left(1 - \frac{B_{02}^{(s+j)}}{B_{00}^{(s+j+1)} + B_{01}^{(s+j+1)}} \right)$$

$$\frac{|[0,1]^{(s+j+1)}|}{B_{00}^{(s+j+1)} + B_{01}^{(s+j+1)}} \leq \rho_{s+j} \left(1 - \frac{B_{01}^{(s+j-2)}}{B_{00}^{(s+j+1)} + B_{01}^{(s+j+1)}} \right).$$

Since additionally $\varepsilon_s = \alpha$ and $\varepsilon_{s-1} = \alpha$ then

$$\frac{|[0,2]^{(s+j+1)}|}{B_{00}^{(s+j+1)} + B_{02}^{(s+j+1)}} = \frac{|[0,1]^{(s+j)}|}{B_{00}^{(s+j)} + B_{01}^{(s+j)}} \leq \rho_{s+j-1}\left(1 - \frac{B_{01}^{(s+j-3)}}{B_{00}^{(s+j)} + B_{01}^{(s+j)}}\right)$$

$$|[1,2]^{(s+j+1)}| \leq \rho_{s+j}(B_{01}^{(s+j+1)} + B_{02}^{(s+j+1)})\left(1 - \frac{B_{01}^{(s+j-1)}}{B_{01}^{(s+j+1)} + B_{02}^{(s+j+1)}}\right).$$

Now we continue the proof of the last theorem.

Note that

$$B_{00}^{(r+1)} + B_{01}^{(r+1)} \leq (t_{r+1} + 2)(B_{00}^{(r)} + B_{01}^{(r)})$$

and if $t_{r+1} \geq 1$ we find

$$B_{00}^{(r+1)} + B_{01}^{(r+1)} \leq 2B_{01}^{(r+1)}.$$

Furthermore if $\varepsilon_{r+1} = \alpha$ also

$$B_{01}^{(r+1)} + B_{02}^{(r+1)} = B_{01}^{(r+1)} + B_{01}^{(r)} \leq (t_{r+1} + 2)(B_{00}^{(r)} + B_{01}^{(r)}).$$

Therefore

$$\kappa_s(x,y) \leq 1 - \frac{1}{2\prod_{i=0}^{5}(t_{s+i-2}(x,y) + 2)}.$$

Define

$$\gamma(x,y) = \begin{cases} \log\left(1 - \frac{1}{2\prod_{i=0}^{5}(t_1(T^{s+i-3}(x,y)+2)}\right) & \text{if} \quad \varepsilon_i = \alpha, t_i \geq 1,\ i = s-2,\ldots,s+3 \\ 0 \quad \text{elsewhere} \end{cases}$$

then a.e.

$$\lim_{N\to\infty} \frac{1}{N}\sum_{m=0}^{N-1} \gamma(T^m(x,y)) = \int_\Delta \gamma(x,y)\,d\mu(x,y) < 0.$$

Since there exists a constant θ such that

$$\limsup_{N\to\infty} \frac{\log B_{01}^{(N)}}{N} \leq \theta < \infty$$

a.e. the result follows by standard methods (see Schweiger 2000).

Remark:

(1) Clearly a more careful analysis of the proof of Lemma 4 and Lemma 5 would give a better value for $\gamma(x,y)$.

(2) Numerical results on the Lyapunov exponents of this algorithm and other 2-dimensional continued fractions can be found in Baldwin 1992b and Baladi & Nogueira 1996.

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