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A Convergence Lemma for the Parry–Daniels Map

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Abstract

Nogueira has shown that the 2-dimensional *Parry–Daniels map* is ergodic. The proof uses the fact that appropriate sequences of cylinders shrink to points. The purpose of this note is to give a proof of this property which shows how the appearance of different types of digits is related to the convergence rate.

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Let $\Sigma = \{x = (x_0, x_1, x_2) : 0 \le x_0, x_1, x_2 \le 1, x_0 + x_1 + x_2 = 1\}$. The 2-dimensional *Parry–Daniels map* $T : \Sigma \to \Sigma$ is defined as follows. Let π be a permutation such $x_{\pi 0} \le x_{\pi 1} \le x_{\pi 2}$ then

$$T(x_0, x_1, x_2) = \left(\frac{x_{\pi 0}}{x_{\pi 2}}, \frac{x_{\pi 1} - x_{\pi 0}}{x_{\pi 2}}, \frac{x_{\pi 2} - x_{\pi 1}}{x_{\pi 2}} \right).$$

Then (Σ, T) is a fibred system (Schweiger 1995, 2000) with the time-1-partition $B(\pi) = \{x \in \Sigma : \pi = \pi(x)\}$. The digits are the six permutations ε, (01), (02), (12), (021), (012). As usual we put $\pi_j = \pi(T^{j-1}x)$,

$j = 1, 2, \ldots$. It is well known that T admits an invariant measure with density

$$h(x) = \frac{1}{x_0(x_0 + x_1)}$$

Daniels (1962) asked if T is ergodic with respect to Lebesgue measure. Parry (1962) proved that the 1-dimensional Parry–Daniels map is ergodic. A further step in this direction was given in Schweiger (1981). Let

$$\Gamma = \{x \in \Sigma : \pi_j(x) = \varepsilon \text{ oder } (01) \text{ for any } j = 1, 2, \ldots\}$$

then $\lambda(\Gamma) > 0$. Therefore T is not conservative. It was reasonable to conjecture that Γ is an *absorbing set* (i.e. for almost all $x \in \Sigma$ there is an $n = n(x)$ such that $T^n x \in \Gamma$). Nogueira (1995) eventually showed that in fact, Γ is an absorbing set and T is ergodic. The proof uses the following result.

Theorem 1. *("Shrinking Lemma"). Let* $\pi_s(x) \in \{(012), (021), (02), (12)\}$ *for infinitely many values of s then* $\lim_{s\to\infty} \operatorname{diam} B(\pi_1, \ldots, \pi_s) = 0$.

We first point out that without an additional condition on the digits $\pi_j, j = 1, 2, \ldots$, the Shrinking Lemma is not generally true. We consider $(\alpha, \beta, \gamma) \in \Sigma, \alpha = -2 + \sqrt{5}, \beta = \frac{7-3\sqrt{5}}{2}, \gamma = \frac{-1+\sqrt{5}}{2}$. Then (α, β, γ) is a fixed point for T. Therefore the segment $\lambda(\alpha, \beta, \gamma) + (1 - \lambda)(0, 0, 1), 0 \leq \lambda \leq 1$, is invariant under T. This shows that

$$\operatorname{diam} B((01), \ldots, (01)) \geq 2\sqrt{5} - 4 > 0.$$

On the other hand geometric considerations strongly suggest the validity of the Shrinking Lemma. The purpose of this note is to give an arithmetic proof of this important fact. We introduce the matrices

$$M(\varepsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad M(01) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$M(12) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad M(021) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M(012) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad M(02) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

If

$$M(\pi_1)\dots M(\pi_s) := \begin{pmatrix} a_{s0} & b_{s0} & c_{s0} \\ a_{s1} & b_{s1} & c_{s1} \\ a_{s2} & b_{s2} & c_{s2} \end{pmatrix}$$

and

$A_s := a_{s0} + a_{s1} + a_{s2}, \quad B_s := b_{s0} + b_{s1} + b_{s2}, \quad C_s := c_{s0} + c_{s1} + c_{s2}$

then the map

$$V_s = V(\pi_1, \dots, \pi_s) : \Sigma \to \Sigma(\pi_1, \dots, \pi_s), \quad V_s x = y$$

$$y_0 = \frac{a_{s0}x_0 + b_{s0}x_1 + c_{s0}x_2}{A_s x_0 + B_s x_1 + C_s x_2}$$

$$y_1 = \frac{a_{s1}x_0 + b_{s1}x_1 + c_{s1}x_2}{A_s x_0 + B_s x_1 + C_s x_2}$$

$$y_2 = \frac{a_{s2}x_0 + b_{s2}x_1 + c_{s2}x_2}{A_s x_0 + B_s x_1 + C_s x_2}$$

maps the simplex Σ onto the cylinder $\Sigma(\pi_1, \dots, \pi_s)$.

We put $\Delta(\pi_1, \dots, \pi_s) := \operatorname{diam} \Sigma(\pi_1, \dots, \pi_s)$. The following lemma is straight-forward.

Lemma 1. *Let $x = (x_0, x_1, x_2), y = (y_0, y_1, y_2), z = (z_0, z_1, z_2)$ be three collinear points, $z = \lambda x + \mu y$, say. Then*

$$\frac{d(V_s x, V_s z)}{d(V_s x, V_s y)} = |\mu| \left| \frac{A_s y_0 + B_s y_1 + C_s y_2}{A_s z_0 + B_s z_1 + C_s z_2} \right|.$$

Lemma 2. *Let $J_0 = V_s(1,0,0)$, $J_1 = V_s(0,1,0)$, $J_2 = V_s(0,0,1)$, $M_1 = V_s\left(\frac{1}{2}, 0, \frac{1}{2}\right)$, $M_2 = V_s\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, $M_0 = V_s\left(0, \frac{1}{2}, \frac{1}{2}\right)$, $Z = V_s\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ then we find the following ratios.*

$$\frac{d(J_2, M_1)}{d(J_0, J_2)} = \frac{A_s}{A_s + C_s}, \quad \frac{d(J_0, M_1)}{d(J_0, J_2)} = \frac{C_s}{A_s + C_s}$$

$$\frac{d(J_0, M_2)}{d(J_1, J_0)} = \frac{B_s}{B_s + A_s}, \quad \frac{d(J_1, M_2)}{d(J_1, J_0)} = \frac{A_s}{B_s + A_s}$$

$$\frac{d(J_1, M_0)}{d(J_2, J_1)} = \frac{C_s}{C_s + B_s}, \quad \frac{d(J_2, M_0)}{d(J_2, J_1)} = \frac{B_s}{C_s + B_s}$$

$$\frac{d(J_0, Z)}{d(J_0, M_0)} = \frac{B_s + C_s}{A_s + B_s + C_s}, \quad \frac{d(M_0, Z)}{d(J_0, M_0)} = \frac{A_s}{A_s + B_s + C_s}$$

$$\frac{d(J_1, Z)}{d(J_1, M_1)} = \frac{C_s + A_s}{A_s + B_s + C_s}, \quad \frac{d(M_1, Z)}{d(J_1, M_1)} = \frac{B_s}{A_s + B_s + C_s}$$

$$\frac{d(J_2,Z)}{d(J_2,M_2)}=\frac{A_s+B_s}{A_s+B_s+C_s}, \quad \frac{d(M_2,Z)}{d(J_2,M_2)}=\frac{C_s}{A_s+B_s+C_s}$$

Lemma 3. *If* $\pi_{s+1}=(02)$ *or* (012) *then*

$$\Delta(\pi_1,\dots,\pi_s,\pi_{s+1})\le\tfrac{3}{4}\Delta(\pi_1,\dots,\pi_s).$$

Proof: Observe that M_1,Z,J_0 are the vertices of $\Sigma(\pi_1,\dots,\pi_s,(012))$ and J_0,Z,M_2 are the vertices of $\Sigma(\pi_1,\dots,\pi_s,(02))$. Since for all $s\ge 1$ the inequalities

$$C_s\le B_s\le A_s$$

are valid we see that

$$\max\left(\frac{C_s}{A_s+C_s},\frac{B_s}{A_s+B_s},\frac{B_s}{A_{s+1}},\frac{C_s}{A_{s+1}},\frac{B_s+C_s}{A_{s+1}}\right)\le\frac{2}{3}.$$

Lemma 4. *If* $\pi_{s+1}=(021)$ *or* (12) *and* $\pi_s\in\{(01),(012),(021),(02)\}$ *then*

$$\Delta(\pi_1,\dots,\pi_s,\pi_{s+1})\le\tfrac{3}{4}\Delta(\pi_1,\dots,\pi_s).$$

Proof: The vertices of $\Sigma(\pi_1,\dots,\pi_s,(021))$ are J_1,Z,M_2 and the vertices $\Sigma(\pi_1,\dots,\pi_s,(12))$ are M_0,Z,J_1. An inspection of the list of ratios shows that the ratio

$$\frac{d(J_1,M_0)}{d(J_1,J_2)}=\frac{C_s}{B_s+C_s}\le\frac{1}{2}$$

is not problematic. A more careful analysis is required for the other ratios. We illustrate the method for $\pi_s=(012),\pi_{s+1}=(021)$. Then

$$\frac{A_s}{A_s+B_s}=\frac{A_{s-1}+B_{s-1}+C_{s-1}}{2A_{s-1}+B_{s-1}+2C_{s-1}}\le\frac{2}{3}$$
$$\frac{A_s+C_s}{A_s+B_s+C_s}=\frac{2A_{s-1}+B_{s-1}+C_{s-1}}{3A_{s-1}+B_{s-1}+2C_{s-1}}\le\frac{3}{4}.$$

Lemma 5. *Let* $\pi_{s+1}=(021)$ *or* (12).

Assume $\pi_j\in\{\varepsilon,(12)\}$ *for* $s-w+1\le j\le s$ *but* $\pi_{s-w}\in\{(01),(012),(021),(02)\}$. *Then*

$$\Delta(\pi_1,\dots,\pi_s,\pi_{s+1}))\le\tfrac{2}{3}\Delta(\pi_1,\dots,\pi_{s-w}).$$

Proof: If $\pi_{s-w}\in\{(01),(012),(021),(02)\}$ then $A_{s-w}\le 2B_{s-w}$.

We note that the cylinder $\Sigma(\varepsilon, \ldots, \varepsilon) = \{x : \pi_j(x) = \varepsilon, 1 \leq j \leq w\}$ has the vertices (0, 0, 1), $Q_w = \left(0, \frac{1}{w+1}, \frac{w}{w+1}\right)$, and

$$R_w = \left(\frac{2}{(w+1)(w+2)}, \frac{2w}{(w+1)(w+2)}, \frac{w}{w+2}\right).$$

Then the cylinder $\Sigma(\varepsilon, \ldots, \varepsilon, (12))$ has the vertices Q_w, Q_{w+1}, and R_{w+1}.

The cylinder $\Sigma(\varepsilon, \ldots, \varepsilon, (021))$ has the vertices Q_w, R_{w+1}, and

$$S_w = \left(\frac{2}{(w+1)(w+4)}, \frac{2}{w+4}, \frac{w(w+3)}{(w+1)(w+4)}\right).$$

We apply Lemma 1 to the map $V_{s-w} : \Sigma \to \Sigma(\pi_1, \ldots, \pi_{s-w})$.

Now assume $\pi_{s+1} = (021)$ or (12). Let $\pi_{s-w} \in \{(012), (02), (021), (01)\}$ and $\pi_j \in \{\varepsilon, (12)\}, s - w + 1 \leq j \leq s$. We first consider the case $\pi_j = \varepsilon, s - w + 1 \leq j \leq s$.

Let $w = 1$. Then the line through $Q_1 = \left(0, \frac{1}{2}, \frac{1}{2}\right)$ and $R_2 = \left(\frac{1}{6}, \frac{2}{6}, \frac{3}{6}\right)$ meets $x_1 = 0$ in the point $T_1 = \left(\frac{1}{2}, 0, \frac{1}{2}\right)$. Therefore

$$\frac{d(Q_1, R_2)}{d(Q_1, T_1)} = \frac{A_{s-1} + C_{s-1}}{A_{s-1} + 2B_{s-1} + 3C_{s-1}} \leq \frac{1}{2}$$

since $A_{s-1} \leq 2B_{s-1}$. We also find

$$\frac{d(Q_1, S_1)}{d(Q_1, J_2)} = \frac{A_{s-1}}{A_{s-1} + 2B_{s-1} + 2C_{s-1}} \leq \frac{1}{2}$$

and

$$\frac{d(S_1, R_2)}{d(S_1, J_2)} = \frac{C_{s-1}}{A_{s-1} + 2B_{s-1} + 3C_{s-1}} \leq \frac{1}{6}.$$

The line through $Q_2 = \left(0, \frac{1}{3}, \frac{2}{3}\right)$ and $R_2 = \left(\frac{1}{6}, \frac{2}{6}, \frac{3}{6}\right)$ meets $x_2 = 0$ in the point $Y_2 = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$. Therefore, again

$$\frac{d(Q_2, R_2)}{d(Q_2, Y_2)} = \frac{2A_{s-1} + B_{s-1}}{2A_{s-1} + 4B_{s-1} + 6C_{s-1}} \leq \frac{2}{3}.$$

Clearly

$$\frac{d(Q_1, Q_2)}{d(Q_1, J_2)} = \frac{1}{3}.$$

Now consider $w \geq 2$. The line through $Q_w = \left(0, \frac{1}{w+1}, \frac{w}{w+1}\right)$ and $R_{w+1} = \left(\frac{2}{(w+2)(w+3)}, \frac{2(w+1)}{(w+2)(w+3)}, \frac{w+1}{w+3}\right)$ meets $x_2 = 0(!)$ in the point $T_w = \left(\frac{2w}{(w+2)(w-1)}, \frac{w^2-w-2}{(w+2)(w-1)}, 0\right)$. Then we find

$$\frac{d(Q_w, R_{w+1})}{d(Q_w, T_w)} \leq \frac{2wA_{s-w} + (w^2 - w - 2)B_{s-w}}{2wA_{s-w} + (2w^2 + 2w)B_{s-w}} \leq \frac{1}{2}.$$

Furthermore, the line through $Q_{w+1} = \left(0, \frac{1}{w+2}, \frac{w+1}{w+2}\right)$ and $R_{w+1} = \left(\frac{2}{(w+2)(w+3)}, \frac{2(w+1)}{(w+2)(w+3)}, \frac{w+1}{w+3}\right)$ meets $x_2 = 0$ in the point $Y_{w+1} = \left(\frac{2}{w+2}, \frac{w}{w+2}, 0\right)$. Then

$$\frac{d(Q_{w+1}, R_{w+1})}{d(Q_{w+1}, Y_{w+1})} \leq \frac{2A_{s-w} + wB_{s-w}}{2A_{s-w} + (2w+2)B_{s-w}} \leq \frac{2}{3}.$$

Last but not least we find

$$\frac{d(Q_w, Q_{w+1})}{d(Q_w, J_2)} = \frac{1}{w+2} \leq \frac{1}{4}.$$

The estimate

$$\begin{aligned} \frac{d(S_w, R_{w+1})}{d(S_w, J_2)} &= \frac{2C_{s-1}}{2A_{s-1} + 2(w+1)B_{s-1} + (w+1)(w+2)C_{s-1}} \\ &\leq \frac{2}{(w+2)(w+3)} \leq \frac{1}{6} \end{aligned}$$

is unconditionally true.

The line through Q_w and R_w meets $x_2 = 0$ in the point $Y_w = \left(\frac{2}{w+1}, \frac{w-1}{w+1}, 0\right)$ and we find as before

$$\frac{d(Q_w, S_w)}{d(Q_w, Y_w)} \leq \frac{d(Q_w, R_w)}{d(Q_w, Y_w)} \leq \frac{2}{3}.$$

In the other case let $\pi_j = \varepsilon, s - w + 1 \leq j \leq t$ and $\pi_{t+1} = (12)$, $t + 1 \leq s$. Since clearly

$$\Delta(\pi_1, \ldots, \pi_{s+1}) \leq \Delta(\pi_1, \ldots, \pi_{t+1})$$

we are back to a situation already considered and we find

$$\Delta(\pi_1, \ldots, \pi_{s+1}) \leq \tfrac{2}{3}\Delta(\pi_1, \ldots, \pi_{s-w}).$$

The only remaining case is $\pi_j \in \{\varepsilon, (12)\}$ for all $j \geq j_0$.

Lemma 6. *Let* $\pi_j \in \{\varepsilon, (12)\}$ *for all* $j \geq 1$. *Then*

$$\Delta(\pi_1, \dots, \pi_s) \ll \frac{1}{\sqrt{s}}.$$

Proof: Let $J_0 = \left(\frac{a_{s0}}{A_s}, \frac{a_{s1}}{A_s}, \frac{a_{s2}}{A_s}\right)$, $J_1 = \left(\frac{b_{s0}}{B_s}, \frac{b_{s1}}{B_s}, \frac{b_{s2}}{B_s}\right)$, and $J_2 = \left(\frac{c_{s0}}{C_s}, \frac{c_{s1}}{C_s}, \frac{c_{s2}}{C_s}\right)$ be the three vertices of $B(\pi_1, \dots, \pi_s)$. Note that $b_{s0} = c_{s0} = 0$ but this fact does not help very much. To estimate the distances $d(J_0, J_1)$, $d(J_1, J_2)$, and $d(J_2, J_0)$ we need estimates for the determinants

$$\begin{aligned} [A,B]_s^j &:= \begin{vmatrix} a_{sj} & b_{sj} \\ A_s & B_s \end{vmatrix}, \\ [B,C]_s^j &:= \begin{vmatrix} b_{sj} & c_{sj} \\ B_s & C_s \end{vmatrix}, \quad j = 0, 1, 2. \\ [C,A]_s^j &:= \begin{vmatrix} c_{sj} & a_{sj} \\ C_s & A_s \end{vmatrix}, \end{aligned}$$

Since the estimate is the same for $j = 0$, 1, 2 we may omit the index j.

Note that for $\pi_{s+1} = \varepsilon$ or (12)

$$\begin{aligned} A_{s+1} &= A_s + B_s + C_s \\ B_{s+1} &= B_s + C_s \end{aligned}$$

but

$$\begin{aligned} C_{s+1} &= C_s \quad \text{if} \quad \pi_{s+1} = \varepsilon \\ C_{s+1} &= B_s \quad \text{if} \quad \pi_{s+1} = (12). \end{aligned}$$

Since $A_1 = 3, B_1 = 2, C_1 = 1$, we get $A_s \leq (s+1)B_s$.

Since $|[B,C]_{s+1}| = |[B,C]_s|$ we find by induction that $|[B,C]_s| \leq 1$. This shows

$$\frac{|[B,C]_s|}{B_s C_s} \leq \frac{1}{B_s} \leq \frac{1}{s+1}.$$

Put $\theta(s) = \max\left(\frac{|[A,B]_s|}{A_s B_s}, \frac{|[C,A]_s|}{A_s C_s}\right)$. Then

$$\begin{aligned} \frac{|[A,B]_{s+1}|}{A_{s+1}B_{s+1}} &\leq \frac{|[A,B]_s| + |[C,A]_s|}{A_{s+1}(B_s + C_s)} \\ &\leq \theta(s) \frac{A_s}{A_s + B_s + C_s} \leq \theta(s) \frac{s+1}{s+2}. \end{aligned}$$

Now assume that $\pi_{s+1} = (12)$. Then

$$\begin{aligned}\frac{|[C,A]_{s+1}|}{A_{s+1}C_{s+1}} &\le \frac{|[A,B]_s| + |[B,C]_s|}{A_{s+1}B_s}\\ &\le \max\left(\frac{|[A,B]_s|}{A_sB_s}\frac{A_s}{A_s+B_s}, \frac{|[B,C]_s|}{C_sB_s}\right)\\ &\le \max\left(\theta(s)\frac{s+1}{s+2}, \frac{1}{s+1}\right).\end{aligned}$$

If $\pi_{s+1} = \varepsilon$ then we find

$$\frac{|[C,A]_{s+1}|}{A_{s+1}C_{s+1}} \le \frac{|[C,A]_s| + |[B,C]_s|}{A_{s+1}C_s}.$$

If $\pi_s = \varepsilon$ we expand further and find

$$\begin{aligned}\frac{|[C,A]_{s+1}|}{A_{s+1}C_{s+1}} &\le \frac{|[C,A]_{s-1}| + 2|[B,C]_{s-1}|}{(A_{s-1}+2B_{s-1}+3C_{s-1})C_{s-1}}\\ &\le \max\left(\frac{|[C,A]_{s-1}|}{A_sC_{s-1}}, \frac{2|[B,C]_{s-1}|}{B_{s-1}C_{s-1}+2C_{s-1}^2}\right)\\ &\le \max\left(\theta(s-1)\frac{s}{s+1}, \frac{2}{B_{s-1}+2}\right)\\ &\le \max\left(\theta(s-1)\frac{s}{s+1}, \frac{2}{s+2}\right).\end{aligned}$$

If $\pi_s = (12)$ we obtain

$$\begin{aligned}\frac{|[C,A]_{s+1}|}{A_{s+1}C_{s+1}} &\le \frac{|[A,B]_{s-1}| + 2|[B,C]_{s-1}|}{(A_{s-1}+3B_{s-1}+2C_{s-1})B_{s-1}}\\ &\le \max\left(\frac{|[A,B]_{s-1}|}{A_sB_{s-1}}, \frac{2|[B,C]_{s-1}|}{2B_{s-1}^2+B_{s-1}C_{s-1}}\right)\\ &\le \max\left(\theta(s-1)\frac{s}{s+1}, \frac{2}{s+2}\right).\end{aligned}$$

Hence $\theta(s+1) \le \max\left(\theta(s-1)\frac{s}{s+1}, \theta(s)\frac{s+1}{s+2}, \frac{2}{s+2}\right)$. It is easy to see that $\theta(s) \le \frac{1}{\sqrt{s+1}}$ satisfies these recursive conditions.

Remark: In fact one can show that

$$E = \{x \in \Sigma : \pi_j(x) = \varepsilon \text{ or } (12) \text{ for all } j \ge 1\} = \{x \in \Sigma : x_0 = 0\}.$$

This follows from the fact that $0 \leq x_0 \leq \frac{1}{s}$ implies

$$(V(\varepsilon)x)_0 = (V(01)x)_0 = \frac{x_0}{1 + 2x_0 + x_1} \leq \frac{1}{s+1}.$$

References

[1] Daniels, H. E.: Processes generating permutation expansions. Biometrika **49**, 139–149 (1962).

[2] Nogueira, A.: The three-dimensional Poincaré continued fraction algorithm. Israel J. Math. **90**, 373–401 (1995).

[3] Parry, W.: Ergodic properties of some permutation processes. Biometrika **49**, 151–154 (1962).

[4] Schweiger, F.: Ergodic Theory of Fibred Systems and Metric Number Theory. Oxford: Clarendon Press, 1995.

[5] Schweiger, F.: Multidimensional Continued Fractions. Oxford: Oxford University Press, 2000.

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